Two-Dimensional Behavior of a Thin
Web on a Roller – Part 2

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ABSTRACT

This paper presents a continuation of work described at the 2009 IWEB conference in a paper titled, "Two-dimensional Behavior of a Thin Web on a Roller" [1]. In that paper (which will be referred to as Part I), linear two-dimensional equations of equilibrium for a thin web on a roller were developed, taking into account cylindrical roller geometry and the effects of friction between the web and roller.

The 2009 paper focused primarily on behavior as the web enters onto a roller. A steady state condition necessary for existence of a stick zone at the entrance was defined. This is particularly useful for predicting slipping on concave and curved-axis spreader rollers.

In Part 2, the following issues are considered.

1. Why is the steady state stick zone always at the roller entry regardless of the direction of microslip? This is one of those innocent questions that stretches the mind. A latex web, operating at large strain, is used to demonstrate that, in the absence of acceleration, the microslip predicted by the capstan equation is a process that propagates upstream from the exit.
2. The lateral entry slip criterion developed in the 2009 paper [1] takes no account of the wrap angle and yet there are clearly situations where wrap matters – even when there is no MD tension difference across the roller. How does this happen?
3. Is there a capstan equation for shear?

The 2D + *w* model, described in a companion paper [2], presented at this conference, will be used to put the cylindrical model on a more rigorous mathematical foundation, using curvilinear coordinates and the nonlinear elasticity equations.

NOMENCLATURE

CD	Cross web direction (*y*-direction)
i, j, k	Unit vector for *x,y,z* coordinate system

$i_{\tilde{x}}, i_{\tilde{y}}, i_{\tilde{z}}$	Unit vectors for $\tilde{x}, \tilde{y}, \tilde{z}$ coordinate system
MD	Machine direction (*x*-direction)
T	Tension, N
R, r	Roller radius, mm
u	Displacement in *x*-direction (or θ-direction), m
v	Displacement in *y*-direction, m
w	Displacement in *r*-direction, m
x, y, z	Cartesian coordinates of relaxed web, m
$\tilde{x}, \tilde{y}, \tilde{z}$	Curvilinear coordinates of deformed web (under stress), m
φ	Coefficient of friction
σ	Stress, N/m^2)
θ	Wrap angle, radians
ψ	Angle of $\tilde{x}$ coordinate relative to x coordinate, radians
Ω	Friction function, N/m^3

Subscripts

avg	Indicates a cross-web average value
d	At downstream roller
r	Indicates relaxed (or reference) state of web
s	Indicates stressed (or current) state of web
u	At upstream roller
x, y, z	Cartesian coordinates of relaxed web

BACKGROUND

The following assumptions are made.

1. The web is in a steady state (no acceleration).
2. The web consists of isotropic material of uniform thickness and width.
3. The coefficient of friction between the web and the roller is uniform and obeys Coulomb laws.
4. Strains are small
5. The material is a linear elastic solid that obeys Hook's law.
6. The effect of air entrainment is not considered.

A review of the standard capstan equation

The diagram in Figure 1 is familiar to everyone who has studied roller traction. In the stick zone, tension is constant and friction insures that the web moves with the roller surface. In the microslip zone, tension is either increasing or decreasing in order to adjust to the value it must reach at the exit. If the roller is a free-turning idler, the stick zone will extend through the entire angle of wrap because there is no tension difference across it. However, if the roller is being braked or driven, the stress in the web must change between the entry and the exit. This is accompanied by changing strain and the only way the web can stretch on the rigid surface is to slip. This slipping is called microslip to distinguish it from the slipping that occurs when the web breaks completely free of the roller. Frictional forces that result from the microslip establish equilibrium between the tension change in the web and the torque τ. The direction of the microslip, relative to the roller surface, can be in either direction, depending on whether the stress is increasing or decreasing as the web moves through the microslip zone.

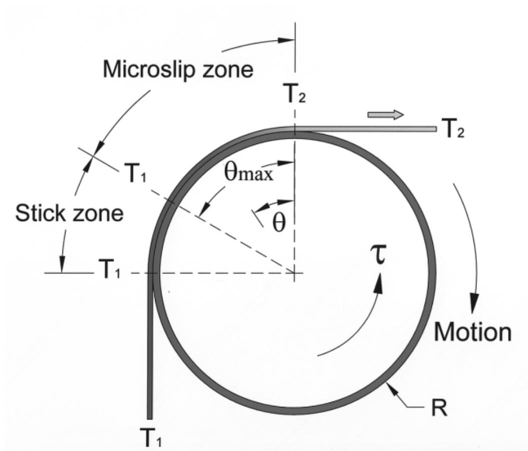

Figure 1 – Web on a roller

In the microslip zone, the behavior of tension is defined by the following one-dimensional differential equation.

$$\frac{dT}{d\theta} = \varphi T \qquad \{1\}$$

This equation is easily integrated. For the arrangement illustrated in Figure 1,

$$T = T_2 e^{\varphi\theta} \qquad \{2\}$$

The angle θ is interpreted in this paper as negative CCW with zero at the exit point of the web. The coefficient of friction φ is will be positive when $T_2 > T_1$ and negative when $T_2 < T_1$. T is assumed to be in units of force and θ in radians. The direction of microslip (direction of increasing strain) depends on whether the roller is driving or braking. It will be positive for braking and negative for driving.

Equation {1} is easily integrated and the maximum arc length is found to be,

$$\theta_{\max} = \frac{1}{\varphi}\ln\left(\frac{T_1}{T_2}\right) \qquad \{3\}$$

This is usually applied in situations where T_1 and T_2 are known and it is desired to know if $\theta_{\max}$ will exceed the angle of wrap.

A CLOSER LOOK AT THE CAPSTAN MODEL

<u>The traction graph and stick zone location</u>

A word of caution: The primary purpose of the following discussion is to demonstrate that while on a roller, much of a web's behavior begins at the exit and *propagates upstream. And, while the discussion is valid insofar as that point is concerned, it applies only to the semi-static situations described here. In a continuously moving web, considerations of energy conservation add another layer of complexity and are not considered.*

Why is the steady state stick zone always at the roller entry regardless of the direction of microslip? This is not a trivial question. The capstan equation was originally derived for machines in which there is no stick zone. For web handling, Brandenburg[3] in 1972, and Dwivedula[4] in 2005 go to considerable pains to justify the nature of the microslip zone. They both use a discrete model consisting of a chain of solid blocks and springs wrapped onto a roller surface to illustrate how a strain change at one end propagates up the chain. Then, they show that by making the elements progressively smaller, equation {4} is produced. In both cases, this is for a static web on a fixed roller. Brandenburg makes an argument for the location of the stick zone based a continuity equation similar to that used in fluid dynamics. Dwividula uses a thermodynamic argument to prove that it can only be at the entry on a rotating roller (in the absence of tension transients) and the conclusion that it is at the entry is implicit in his discussion of the microslip zone.

Figure 2 – Microslip experiment on a static roller

A helpful way to study the question of stick zone location is to make a plot of tension versus distance along the arc as shown in Figure 3. For sake of illustration an arrangement like Figure 2 will be considered. The roller is assumed to be initially locked and a belt with a weight at each end is draped over it in a 180 degree wrap. At what will later become the entry, a weight of 1 Newton is attached. At the exit end, the weight is 2 Newtons. Under these circumstances, two microslip zones will develop and they will intersect at a point where the tensions on both sides simultaneously satisfy equation {2}. It should be noted that equation {1} is a limiting condition that applies while a web is moving relative to the roller. On a fixed roller like the one shown here, microslip zones will be *active* only while they are forming. So, the web will quickly settle into a state of equilibrium and the microslip will become *inactive*.

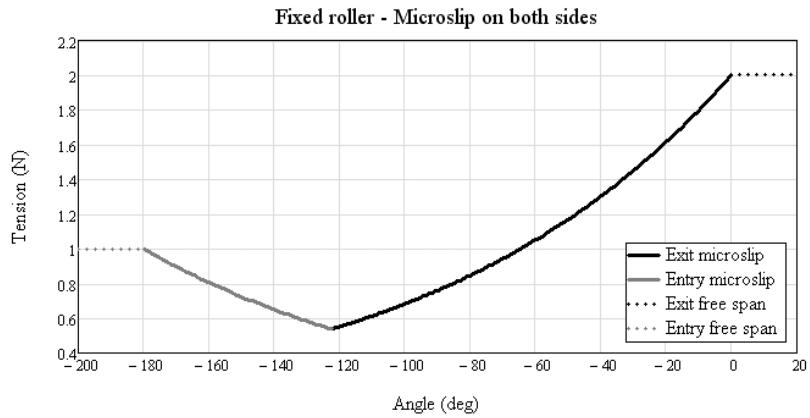

Figure 3 – Microslip zones on fixed roller

Now, imagine that the roller is unlocked and slowly rotated by hand, 10 degrees toward the exit.

Note: A continuously moving web doesn't behave this way. In this setup (described in Figure 6) the roller is never free to respond to the tension differential across the roller. It is presented only because it illuminates the underlying physics.

<u>The graph that can't happen</u>

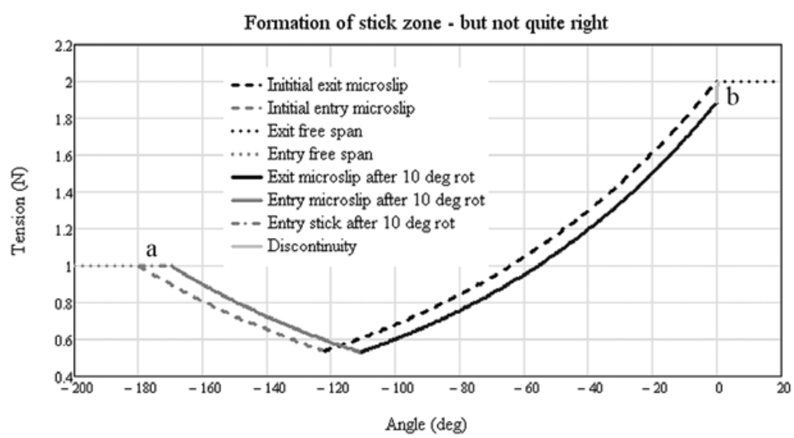

Figure 4 – A dilemma at (b)

The web is in equilibrium with friction forces on the roller. So, an initial assumption might be made that all of it moves with the roller as shown in Figure 4. At point (a), there is no problem with this assumption. The web enters onto the roller at the same tension it had in the free span. So, in segment (a), the derivative of T with respect to θ is zero and that's allowed by equation {1} because there is no relative motion between the web and roller. But, there is a problem at point (b). As the web moves off the roller into the free span, it will have to make a rapid change in tension. This would require the derivative of T with respect to θ to become very large – larger than the limit of φT imposed by equation {1}. So, something else must happen there.

The graph that is possible

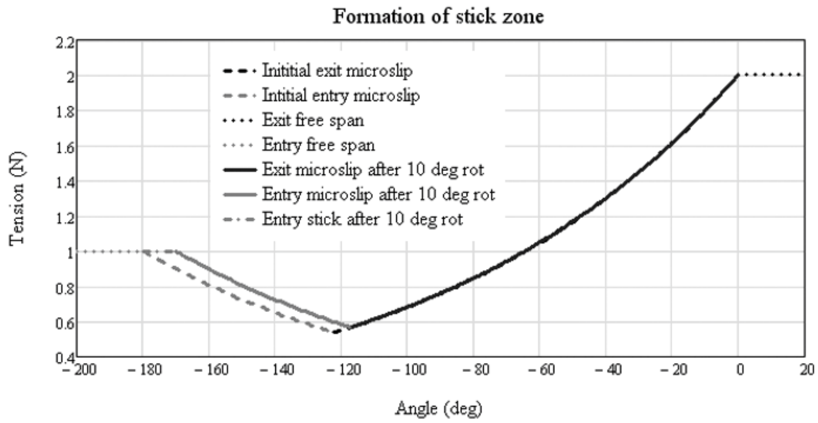

Figure 5 – What really happens

Figure 5 shows what has to happen. From the moment the roller begins rotating, a tiny bit of the web, immediately adjacent to the exit point, experiences a small increment of tension sufficient to bring it up to the exit tension. That creates a disturbance that travels upstream toward the entry at the speed of sound [5], causing each point, in turn, to change its tension as required by equation {1}. This process stops at the point where the new exit segment meets the displaced entry segment. As the roller continues to rotate, this process continues until the entry microslip segment is completely consumed and replaced by a segment at constant tension. The fact that the two curves intersect as shown in the diagram makes sense when it is remembered that 1) for a given coefficient of friction their shapes are determined only by the entry and exit tensions, which aren't changing and 2) only the entry segment is moving with the roller.

An experiment

The general features described above were qualitatively confirmed in a very rough way by a simple experiment with the apparatus shown in Figure 2. Strains were made large (as much as 0.5) to facilitate measurement. Stick-slip was the biggest source of error. Efforts were made to help it with some cautious banging on the machine frame. However, the data suggest that a considerable amount remained. The data shown here was smoothed using a running average with a coefficient of 0.5. Significant Poisson contraction was evident and may have affected the assumption of uniaxial stress.

The belt was made of latex, 1 inch wide (25.4 mm) wide and 0.0045 inch (0.11 mm) thick. Test weights were 1.027 N and 1.699 N. The friction coefficient was measured as 0.62. Rather than make an independent measurement of the stress-strain relationship of the latex, the strain data was scaled to make the value at the exit end equal to the numerical value of the test weight.

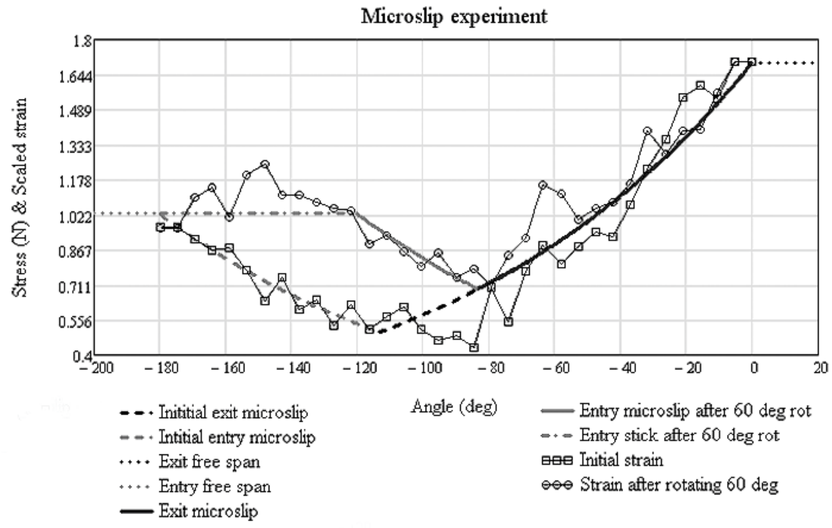

Figure 6 – Microslip experiment

The scale on the latex was made manually using a metal scale, with marks every 1/10 inch (2.54 mm). There was enough variation to require calibration. It was measured, while relaxed and flat on a table, using a precision scale and a magnifying glass. When on the roller, the strain was measured by taping a paper strip next to the latex scale and placing a pencil mark on the paper opposite each scale marking. This was the second largest source of error – on the order of 0.02 inch (0.5 mm). Then, the paper strip was taken off the roller, laid flat, and measured in the same way as the latex scale.

Everything considered, the data fits surprisingly well. It's good enough to provide at least modest assurance that the reasoning is correct.

Microslip created at a roller immediately upstream of a misaligned roller

Anyone who has worked with a web guide is familiar with the limitations imposed by the upstream moment it creates. In extreme cases, the slack edge will lift off the roller. In the photo below, grid distortion can be seen on the web before it has reached the exit, revealing the presence of two microslip zones. On the tight side, the web is slipping forward, relative to the roller surface, and on the low-tension side it is slipping backward. Something to bear in mind is that this kind of microslip could cause loss of traction at the entry side well before its effects are otherwise evident. This is latex, so it is quite tolerant. But, stiff materials such as paper and PET will be very susceptible. Note also that, on a free-turning idler, the stresses will distribute themselves so that there is no net torque on the roller.

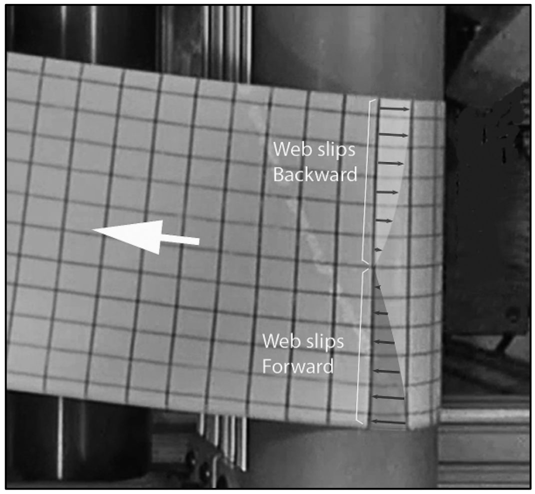

Figure 7 – Microslip due to non-uniform stress upstream of a misaligned roller

It is possible that something similar to this may account for the small discrepancies seen in cambered web models.

OTHER WAYS TO SLIP

In Part I, a criterion was developed for the onset of lateral slipping at the entry to a roller. This was done with concave rollers in mind, where there is very small turning torque and the web exerts zero net lateral force. It assumes steady state tension and takes no account of the amount of wrap. Wrap obviously matters, though. If a roller is driving or braking, θ_{max} in equation {3} may exceed the angle of wrap. Or, if there is a net lateral force at the entry, some amount of wrap will be necessary to avoid slipping. So, the entry slipping criterion is a necessary condition for the existence of a stick zone but it is not sufficient. First the capstan equation will be rearranged and then the entry slipping criterion will be reviewed.

First, a rearrangement of the capstan equation

Equation {1} will be recast in terms of arc length instead of angle. The second dimension of width will also be added; although any lateral effects will be ignored for the time being (that is actually an implicit assumption in the derivation of the capstan equation). Also, tension will be assumed to be in units of stress. Dividing both sides by R, and changing the symbol for tension to σ (to remind us it is in units of stress), the equation for the microslip zone now looks like this.

$$\frac{d\sigma_{xx}}{dx} = \varphi \frac{|\sigma_{xx}|}{R} \tag{4}$$

Arc length is equal to $R\theta$ and is represented by x. Note that this means x will be defined in the same way as theta – zero at the exit point and negative in the CCW direction. Also, the sign convention previously established for φ is used. The $|\sigma_x|/R$

term is stress per unit of circumference. This says that for any point in the microslip zone, the rate of change of stress with distance along the arc is equal to the product of curvature, stress and the coefficient of friction. It should be noted that the quantity on the left is the spatial rate of change of stress and has units of stress per unit of distance as does the term on the right. So, the term on the left will be referred to as the *stress rate* and the term on the right will be called the *friction rate* where it is understood that rate refers to the spatial rate of change. Both terms originally included a common factor of (dA)(dx) that has been canceled out. So, this is actually a condition of equilibrium between the forces of elasticity and friction for an infinitesimal element.

For values of x between zero and $\theta_{\max}$, the tension will be,

$$T = T_2 e^{\varphi x/R} \qquad \{5\}$$

A recap of the argument for the entry slipping criterion

Before reaching the roller, all of the stresses in the web are in equilibrium. However, when the web arrives on the roller and assuming it isn't slipping, there can be no variation of stress in the x-direction. As it enters onto the roller, the web can be imagined as a series of narrow strips parallel to the roller axis. If the web is in a steady state, each strip enters onto the roller in the same state as the one just before it. So, in the stick zone, the derivatives of stress with respect to x disappear and are replaced by forces of friction.

To keep the discussion simple, assume that the linear equations of elasticity can be used (the nonlinear version will be developed later). In open spans they are,

x-direction $$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \qquad \{6\}$$

y-direction $$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \qquad \{7\}$$

After entering onto the roller, the equations of equilibrium become,

x-direction $$\frac{\partial \sigma_{xy}}{\partial y} = friction_x \qquad \{8\}$$

y-direction $$\frac{\partial \sigma_{yy}}{\partial y} = friction_y \qquad \{9\}$$

and,

$$\left.\frac{\partial \sigma_{xy}}{\partial x}\right|_{at\,entry} = -friction_x \qquad \left.\frac{\partial \sigma_{xx}}{\partial x}\right|_{at\,entry} = -friction_y \qquad \{10\}$$

where the derivatives with respect to x have been replaced by components of a friction vector. All of the terms are in units of spatial rates of change of stress and can be treated as vectors. So, the vector sum of the friction terms must be less than or equal to the vector sum of the terms on the left.

$$\text{Maximum spatial rate of change of friction} = \quad \varphi\frac{|\sigma_{xx}|}{r} \tag{11}$$

where φ is the dynamic coefficient of friction and *r* is the radius of the roller. The entry slip criterion thus becomes,

$$\sqrt{\left(\frac{\partial\sigma_{xy}}{\partial x}\right)^2_{at\,entry}+\left(\frac{\partial\sigma_{xx}}{\partial x}\right)^2_{at\,entry}} \le \varphi\frac{|\sigma_{xx}|}{r} \tag{12}$$

A capstan equation for shear

The capstan equation, described at the beginning of this paper, depends for its credibility on the fact that MD stress is dominant in webs and in many cases, uniform MD stress can be assumed. But, we know from experience that lateral stresses can have big effects on traction, particularly shear stress. A small step toward a full 2D model would be to consider a capstan equation for shear. Although this will require a number of questionable assumptions, it's instructive. Such an equation might look like this.

$$\frac{d\sigma_{xy}}{dx} = \varphi\frac{|\sigma_{xx}|}{R} \tag{13}$$

Consider a misaligned roller. We know that it has a parabolic shear stress profile (it has to be zero at the unsupported edges). But, for the moment we will assume that that it has a uniform width-wise distribution with the same average value as the parabolic profile. If the web has adequate traction, this stress will be transported through the wrap to the exit and at that point it faces the same situation illustrated in Figure 4 for the MD tension. The shear stress in the downstream span will adjust to conditions there (generally zero shear) in a very short distance. This distance may be too short to allow $d\sigma_{xy}/d\theta$ to satisfy equation {13}. But as soon as increments of web start showing up with new values of σ_{xy}, they experience forces that are on the threshold of slipping and are tugged into place. These increments of displacement are accompanied by stress disturbances that travel upstream to displace neighboring increments in a similar way. The process ends when σ_{xy} equals the entry value in the stick zone and all points in the "lateral" microslip zone fit equation {13}.

Equation {13} is easily integrated. If : σ_{xy1} is the entrance value, σ_{xy2} the exit value, *x* negative for CCW θ and φ positive when $\sigma_{xy2} > \sigma_{xy1}$. The result is,

$$\sigma_{xy} = \varphi\frac{|\sigma_{xx}|}{R}x + \sigma_{xy2} \tag{14}$$

And,

$$\theta_{\max}R = \left(\sigma_{xy1} - \sigma_{xy2}\right)\frac{R}{\varphi|\sigma_{xx}|} \tag{15}$$

With a little manipulation it can be shown that {14} is equivalent to,

$$F_{xy} = \varphi \theta_{\max} F_x \qquad \{16\}$$

where F_{xy} is the maximum lateral force that can be supported across the roller, F_x is the MD tension in units of force and $\theta_{\max}$ is the angle of wrap in radians. This agrees with guidelines currently in use [6]. However, it should be remembered that this began with an assumption of a uniform distribution of shear stress. *So, if this analysis is correct, it is possible to meet the criterion in {16} and have lateral microslip in the center portion of the web on an idler. This could be a factor in the onset of wrinkling.*

To get a little feeling for {15}, values of shear were calculated for a misaligned roller using an FEA model.

Length	60 inches (1.52 m)
Width	40 inches (1.02 m)
Thickness	0.001 inch (25.4 microns)
Tension	0.5 pli [500 psi (3.5 MPa)]
Modulus	50,000 psi (0.34 GPa)
Poisson ratio	0.35
Roller radius	3 inches (76 mm)
Roller Angle	1 deg (0.017 radian)
Coefficient of friction	0.25

The average shear stress is 53.4 psi (0.37 MPa). So, $\theta_{\max}$ is 24.5 degrees (0.43 radian)

It's worth noting that the FEA analysis indicated that the entry slip criterion defined in {12} was barely met.

One of the original goals for this paper was to photograph visual evidence of lateral microslip at the exit of a misaligned roller, using a latex web operating with large strains. Considerable effort was made without success. The amount of shear stress at a misaligned roller is a small fraction of the MD stress. So, it's possible that the effect is simply too subtle to observe with the unaided eye, even at large strains – at least in the case of a misaligned roller.

THE 2D + *W* EQUATIONS FOR A WEB ON A ROLLER IN CYLINDRICAL CURVILINEAR COORDINATES

For an introduction to the 2D + *w* model, please refer to, "The Use of Conservation of Mass in Modeling Lateral Behavior in Moving Webs" [2], presented at this conference. For a cylinder with *y*-axis symmetry, radius *r*, and azimuth angle θ, the coordinate transformations are,

$$x = r\cos(\theta) \qquad y = y \qquad z = r\sin(\theta) \qquad \{17\}$$

When using nonlinear elasticity theory, new coordinate lines, which are generally curved, but orthogonal , apply to the web in its stressed state. These contain the points which, before deformation, were located on lines parallel to the corresponding coordinate axes *x*, *y* and *z*. In the case of plane stress, *z* is assumed to be normal to the *x*-*y* plane. For a two-dimensional problem, you can imagine that if a rectangular grid is inscribed on the object in the relaxed state, it then becomes a curvilinear coordinate system for the object after it is deformed by stress. The subscripts $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are used to indicate these coordinates of the stressed web. The unit vectors representing the coordinate directions will be designated $i_{\tilde{x}}$, $i_{\tilde{y}}$ and $i_{\tilde{z}}$. If the strains are small, this new coordinate system can be considered to be mutually orthogonal and will generally be rotated relative to the *x*, *y*, and *z* axes by an amount that will vary depending on location. When changing to a curvilinear coordinate system such as $r, \theta \, and \; y$, the deformed curvilinear coordinates for the stressed web will be denoted using tildes in the same manner , $\tilde{\theta}, \tilde{y} \, and \; \tilde{r}$.

The displacement variables in the cylindrical coordinate system will be *u* for the θ-direction, *v* for the *y*-direction and *w* for the *r*-direction.

It is important to keep in mind that although all of the derivatives with respect to *r* disappeared, the role of *r* in defining the geometry must remain and in this particular model (a membrane on a roller), wherever the radius *r* appears alone it is retained as a constant. To make that fact easier to remember, *r* will be capitalized.

Unlike the equations for a torus, described in reference [2], the transformations for cylindrical coordinates are sufficiently simple that quantities such as $\left(\frac{1}{2}e_{\theta y} - \omega_r\right)$ can be algebraically reduced to single terms and intermediate quantities like $\widehat{a_\theta a_\theta}$ (used in [2]) can be avoided.

The Lamé coefficients for a cylinder are,

$$H_\theta = R \qquad H_y = 1 \qquad H_r = 1 \tag{18}$$

The parameters corresponding to e_{xx}, e_{yy}, e_{xy}, e_{xz}, e_{yz} ω_x, ω_y and ω_z are:

$$e_{\theta\theta} = \frac{1}{R}\frac{\partial u}{\partial \theta} \tag{19}$$

$$e_{yy} = \frac{\partial v}{\partial y} \tag{20}$$

$$e_{\theta y} = \frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial y} \tag{21}$$

$$e_{\theta r} = \frac{1}{R}\left(\frac{\partial w}{\partial \theta}\right) \tag{22}$$

$$e_{yr} = \frac{\partial w}{\partial y} \tag{23}$$

$$\omega_\theta = \frac{1}{2}\frac{\partial w}{\partial y} \tag{24}$$

$$\omega_y = \frac{1}{2}\frac{1}{R}\frac{\partial w}{\partial \theta} \tag{25}$$

$$\omega_r = \frac{1}{2}\left(\frac{1}{R}\frac{\partial v}{\partial \theta} - \frac{\partial u}{\partial y}\right) \tag{26}$$

The equations of equilibrium for the θ-direction, *y*-direction and *r*-direction are, in that order,

$$\frac{1}{R}\frac{\partial}{\partial\theta}\left[\sigma_{\tilde{\theta}\tilde{\theta}}\left(1+e_{\theta\theta}\right)+\sigma_{\tilde{\theta}\tilde{y}}\frac{\partial u}{\partial y}\right]+\frac{\partial}{\partial y}\left[\sigma_{\tilde{y}\tilde{\theta}}\left(1+e_{\theta\theta}\right)+\sigma_{\tilde{y}\tilde{y}}\frac{\partial u}{\partial y}\right]$$
$$+\frac{1}{R}\left(\sigma_{\tilde{\theta}\tilde{\theta}}\frac{1}{R}\frac{\partial w}{\partial\theta}+\sigma_{\tilde{\theta}\tilde{y}}\frac{\partial w}{\partial y}\right)=0 \qquad \{27\}$$

$$\frac{1}{R}\frac{\partial}{\partial\theta}\left[\sigma_{\tilde{\theta}\tilde{\theta}}\frac{1}{R}\frac{\partial v}{\partial\theta}+\sigma_{\tilde{\theta}\tilde{y}}\left(1+e_{yy}\right)\right]+\frac{\partial}{\partial y}\left[\sigma_{\tilde{y}\tilde{y}}\left(1+e_{yy}\right)+\sigma_{\tilde{y}\tilde{\theta}}\frac{1}{R}\frac{\partial v}{\partial\theta}\right]=0 \qquad \{28\}$$

$$\frac{1}{R}\frac{\partial}{\partial\theta}\left(\sigma_{\tilde{\theta}\tilde{\theta}}\frac{1}{R}\frac{\partial w}{\partial\theta}+\sigma_{\tilde{\theta}\tilde{y}}\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(\sigma_{\tilde{y}\tilde{\theta}}\frac{1}{R}\frac{\partial w}{\partial\theta}+\sigma_{\tilde{y}\tilde{y}}\frac{\partial w}{\partial y}\right)$$
$$-\frac{1}{R}\left[\sigma_{\tilde{\theta}\tilde{\theta}}\left(1+e_{\theta\theta}\right)+\sigma_{\tilde{\theta}\tilde{y}}\frac{\partial u}{\partial y}\right]=0 \qquad \{29\}$$

The last group of terms in the *r*-direction equation is responsible for the radial pressure between the web and the roller.

Equations {27} through {29} represent a membrane, that in its relaxed state is shaped like the surface of a cylinder (with its axis of symmetry on the *y*-axis), and *w* is any displacement from that surface caused by a load applied to it. Now, if we want this to represent a web that is being pressed against a roller, *w* must be set to zero and all the terms involving *w* will go to zero. That will remove the third group of terms from equation {27} and both the first and second groups from equation {29}. Intuitively, it seems that the last equation should disappear entirely. But, the last group is not affected by setting *w* to zero. This is because we are assuming that the web is being pressed against a roller and that the roller is providing a reaction pressure to support it. However, nothing in the derivation of the equations presumed the existence of such a reaction. So, the existence of the roller cancels out the third group in {29}. It's going to reappear, though, in the first two equations as part of friction terms.

Finally, all of the products $R\partial\theta$ can be replaced by a new variable which will be called *x*. Making this change and setting *w* equal to zero, the equilibrium equations for a web on a roller with friction are,

$$\frac{\partial}{\partial x}\left[\sigma_{\tilde{x}\tilde{x}}\left(1+e_{xx}\right)+\sigma_{\tilde{x}\tilde{y}}\frac{\partial u}{\partial y}\right]+\frac{\partial}{\partial y}\left[\sigma_{\tilde{y}\tilde{x}}\left(1+e_{xx}\right)+\sigma_{\tilde{y}\tilde{y}}\frac{\partial u}{\partial y}\right]=\Omega_x \qquad \{30\}$$

$$\frac{\partial}{\partial x}\left[\sigma_{\tilde{x}\tilde{x}}\frac{\partial v}{\partial x}+\sigma_{\tilde{x}\tilde{y}}\left(1+e_{yy}\right)\right]+\frac{\partial}{\partial y}\left[\sigma_{\tilde{y}\tilde{y}}\left(1+e_{yy}\right)+\sigma_{\tilde{y}\tilde{x}}\frac{\partial v}{\partial x}\right]=\Omega_y \qquad \{31\}$$

The terms Ω_x and Ω_y are friction functions.

The strains corresponding to $\varepsilon_{\tilde{x}\tilde{x}}$, $\varepsilon_{\tilde{y}\tilde{y}}$ and $\varepsilon_{\tilde{x}\tilde{y}}$ are:

$$\varepsilon_{\tilde{x}\tilde{x}}=e_{xx}+\frac{1}{2}\left[e_{xx}{}^2+\left(\frac{\partial v}{\partial x}\right)^2\right] \qquad \{32\}$$

$$\varepsilon_{\tilde{y}\tilde{y}} = e_{yy} + \frac{1}{2}\left[e_{yy}{}^{2} + \left(\frac{\partial u}{\partial y}\right)^{2} \right] \qquad \{33\}$$

$$\varepsilon_{\tilde{x}\tilde{y}} = e_{xy} + e_{xx}\frac{\partial u}{\partial y} + e_{yy}\frac{\partial v}{\partial x} \qquad \{34\}$$

The stresses are:

$$\sigma_{\tilde{x}\tilde{x}} = \frac{E}{1-\mu^2}\left(\varepsilon_{\tilde{x}\tilde{x}} + \mu\varepsilon_{\tilde{y}\tilde{y}}\right) \qquad \{35\}$$

$$\sigma_{\tilde{y}\tilde{y}} = \frac{E}{1-\mu^2}\left(\varepsilon_{\tilde{y}\tilde{y}} + \mu\varepsilon_{\tilde{x}\tilde{x}}\right) \qquad \{36\}$$

$$\sigma_{\tilde{x}\tilde{y}} = \frac{E}{2\left(1+\mu\right)}\left(\varepsilon_{\tilde{x}\tilde{y}}\right) \qquad \{37\}$$

The problem now remains to determine the friction functions Ω_x and Ω_y. In the microslip zone,

$$\sqrt{\Omega_x{}^2 + \Omega_y{}^2} = \varphi\frac{1}{R}\left|\sigma_{\tilde{x}\tilde{x}}\left(1+e_{xx}\right) + \sigma_{\tilde{x}\tilde{y}}\frac{\partial u}{\partial y}\right| \qquad \{38\}$$

And in the stick zone,

$$\sqrt{\Omega_x{}^2 + \Omega_y{}^2} < \varphi\frac{1}{R}\left|\sigma_{\tilde{x}\tilde{x}}\left(1+e_{xx}\right) + \sigma_{\tilde{x}\tilde{y}}\frac{\partial u}{\partial y}\right| \qquad \{39\}$$

Also in the stick zone, (following the same reasoning as for equation {10} in the earlier discussion of the entry slip criterion), the following relationships will hold.

$$\begin{aligned} \Omega_x &= -\frac{\partial}{\partial x}\left(\sigma_{\tilde{x}\tilde{x}}\left(1+e_{xx}\right) + \sigma_{\tilde{x}\tilde{y}}\frac{\partial u}{\partial y}\right)\Bigg|_{at\ entry} \\ and \quad \Omega_y &= -\frac{\partial}{\partial x}\left(\sigma_{\tilde{x}\tilde{x}}\frac{\partial v}{\partial x} + \sigma_{\tilde{x}\tilde{y}}\left(1+e_{yy}\right)\right)\Bigg|_{at\ entry} \end{aligned} \qquad \{40\}$$

Equations {38} through {40} are not sufficient to create a working model. Something more is needed to define the direction of microslip. But, this is getting close. The rest will have to wait for Part 3.

Validation of the model

This model has been coded and successfully tested for cases in which there is no friction. In one instance, a frictionless roller surface was assumed with $\sigma_{\tilde{x}\tilde{x}}$ set to 1000 psi in a load boundary condition at the downstream end. The web was fixed in the *x*-direction at the upstream end and left free in the *y*-direction at both ends. The edges were left free.

To eliminate rigid body motion, the integral of *v* on the perimeter was set to zero. Intuitively, one would expect that under these conditions, the web would show no variation in *x*-direction stress and *y*-direction stress would be zero. That was the case

CONCLUSIONS

Location of the stick zone

Changes in tension in the microslip zone propagate from the exit toward the entry and this explains why the stick zone is at the entry.

If the hypothesis about formation of a shear microslip zone is correct:

Lateral stresses may propagate from back to front in the same manner as longitudinal stress and either contribute to microslip zones or create their own, even on an idler.

Because of the presumed linear relationship between shear-microslip and wrap angle, the amount of wrap necessary to prevent slipping can be calculated using current methods. However there may often be situations in which the lateral profile of shear stress, which is usually non-uniform, causes its microslip zone to penetrate into the line of entry causing loss of traction over only the central part of the web. This could aggravate wrinkling problems.

Progress toward a complete model

Significant progress has been made toward a full 2D model of traction on a roller. The 2D+*w* model with cylindrical curvilinear coordinates incorporates the features of nonlinear elasticity and the terms for radial pressure appear in the results as a natural consequence of a well-established formal procedure. And a better understanding of the nature of microslip has been gained through testing and analysis.

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