Propagation of Longitudinal Tension in a Slender Moving Web

IWEB 99

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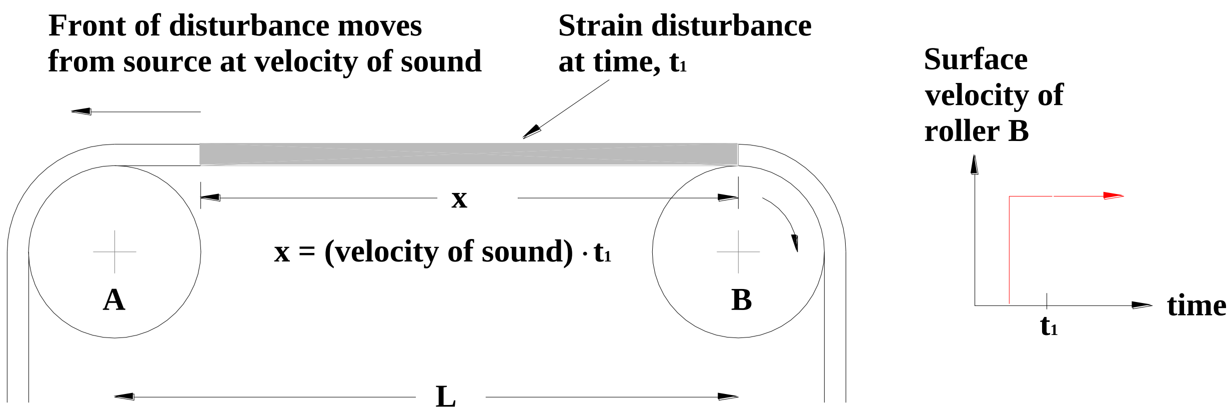
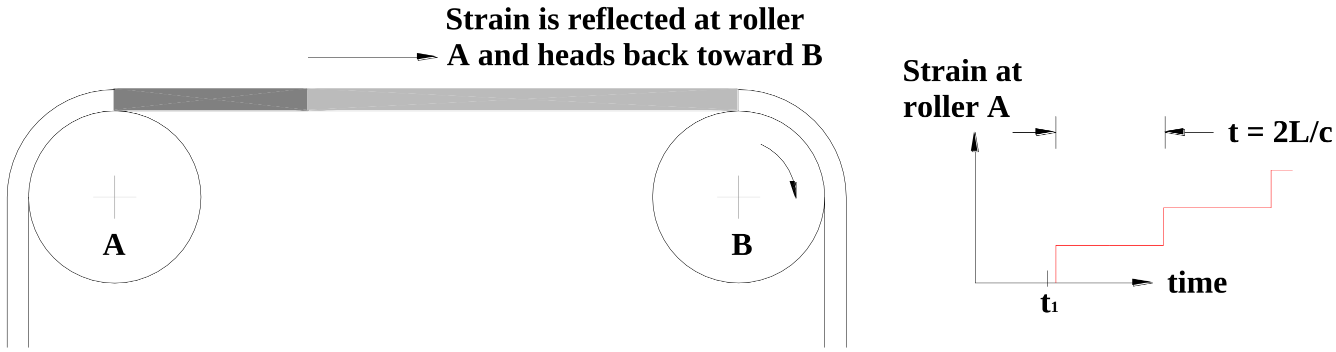
Essex Systems

Strain propagation in a web that isn't moving





Current models

- Current models don’t include propagation effects.
- Processes, so far, have not required it.
- In polymers sound travels at 470 to 1,800 m/s.
- It would take only .002 to .006 sec to travel a 3 m span.
- The transport time at 2.5 m/s would be 1.2 sec.

Then, why consider propagation?

- There are ways propagation interacts with transport motion to affect even slow processes.
  - Propagation of strain discontinuities.
  - Amplification of repetitive disturbances.
  - Damping of a solitary disturbance.
- Line speeds are increasing. In a paper line running at 50 m/s the transport time for 3 m is only .06 sec.
- It will help future work on out-of-plane motion, viscoelasticity and aerodynamics.

Earlier work

- Many papers have been published on out-of-plane vibrations in “traveling strings”.
- Some papers treated the longitudinal tension variation that accompanied the out-of-plane motion.
- Nothing seems to have been published on longitudinal tension propagation as a principal feature of solid material transport.

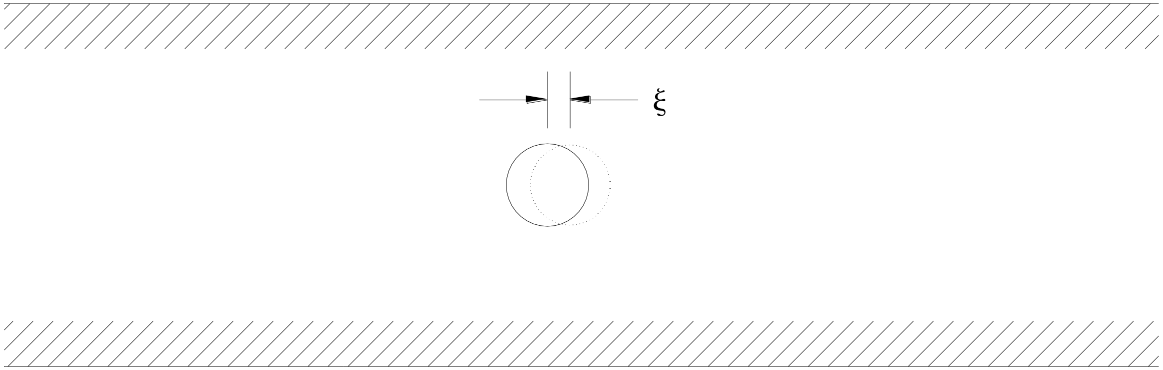
Assumptions

- Both rollers are driven with perfectly controlled speeds.
- Coulomb friction exists between the web and the rollers.
- On the roller, the web obeys the capstan relationship.

Assumptions (Cont.)

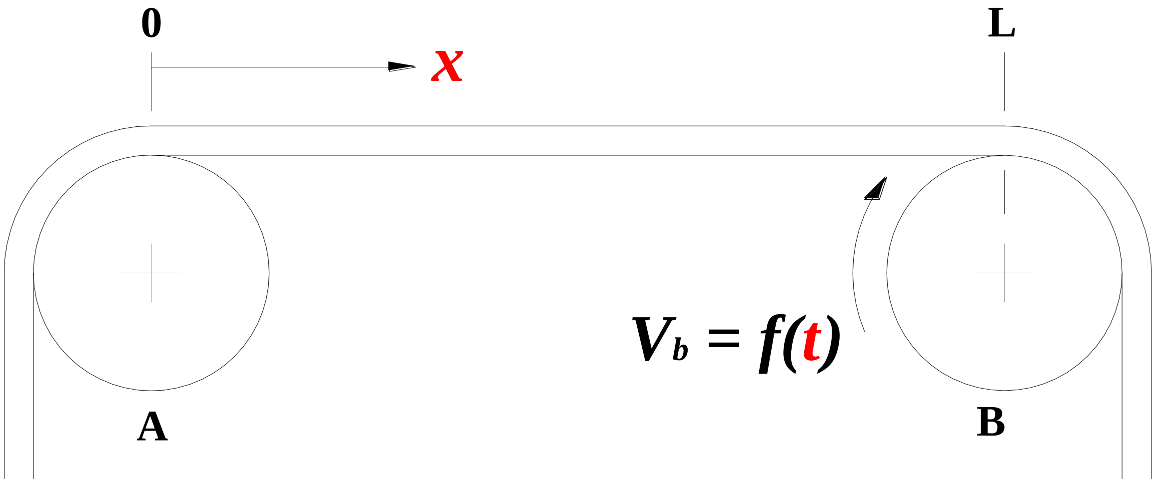
- The web is uniform in its relaxed state.
- The web is elastic (obeys Hooke's law).
- The web is perfectly flexible in the transverse direction.

The dependent variable



ξ is the displacement of a web particle caused by material deformation plus the displacement caused by the transport velocity of the web.

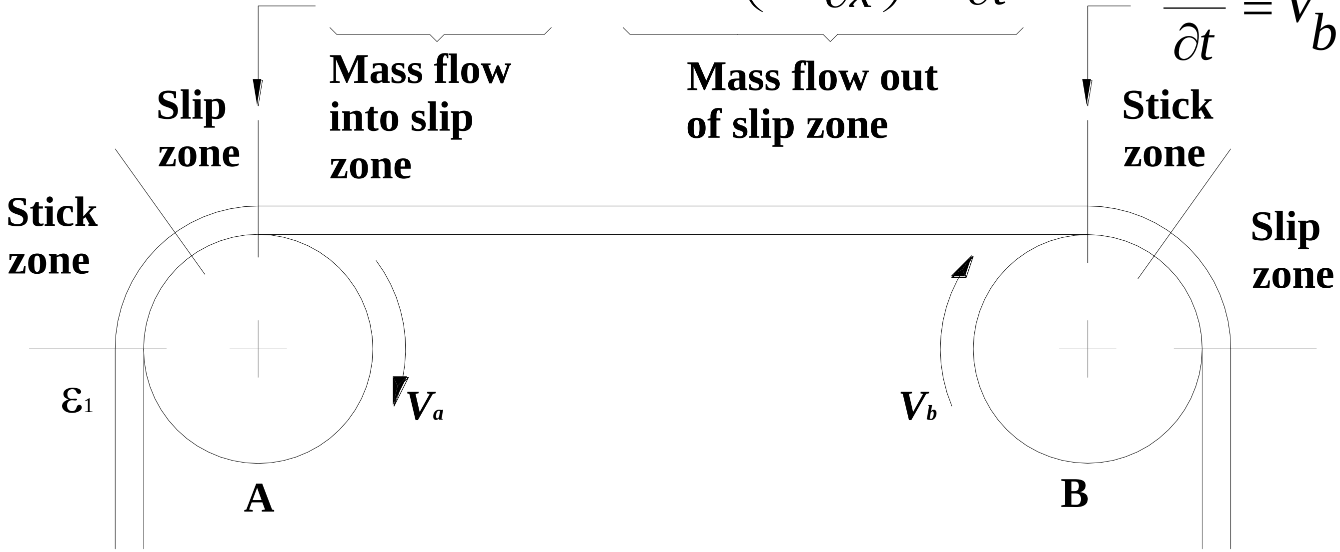
Independent variables



The boundary conditions

$$\rho_o(1+\varepsilon_1)^{-1}V_a = \rho_o\left(1+\frac{\partial\xi}{\partial x}\right)^{-1}\frac{\partial\xi}{\partial t}$$

$$\frac{\partial\xi}{\partial t} = V_b$$



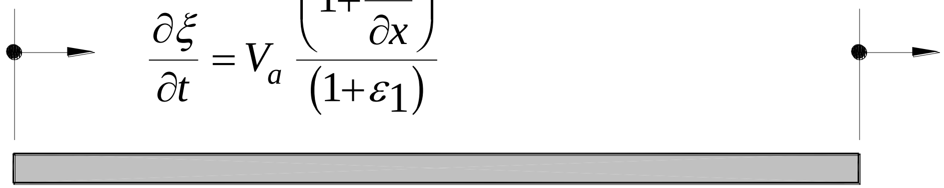
The PDE Model

Boundary Condition I

$$\frac{\partial \xi}{\partial t} = V_a \frac{\left(1 + \frac{\partial \xi}{\partial x}\right)}{\left(1 + \varepsilon_1\right)}$$

Boundary Condition II

$$\frac{\partial \xi}{\partial t} = V_b$$



$$\underbrace{\rho_0 \frac{\partial^2 \xi}{\partial t^2}}_{\text{Inertia}} = \underbrace{E \frac{\partial^2 \xi}{\partial x^2}}_{\text{Elasticity}} \quad \textbf{P.D.E.}$$

Inertia **Elasticity**

It looks like all the pieces are here for a solution. Unfortunately, this isn't the right problem. The web and its boundaries are going to move off to the right with a speed approximately equal to the transport velocity. It's like a flying carpet. Something more is needed.

The Euler Description

- The Euler description is commonly used for problems in moving media.
- It treats the motion of the medium as a field in which the velocity is known at all times and positions.
- Then, the equations describing the physical laws can be modified to apply to a succession of material particles passing a fixed point.
- The technique is commonly used in fluid mechanics where it is called the Material Derivative.

The Euler description (Cont.)

- To avoid confusion, the description we have been using until now is given its own name, the Lagrange description. This is the familiar Newtonian perspective in which quantities, such as acceleration and force, are assumed to be associated with bodies that are moving relative to the observer.
- Note, that neither description has anything to do with moving coordinate systems. Moving coordinate systems can be used to good advantage in problems like this, as shown by Miranker [10]. But, that is a different technique.

The transformation equations

$$\underbrace{\frac{\partial \xi_L}{\partial t}}_{\textbf{Total}} = \underbrace{V_i}_{\textbf{Transport motion}} + \underbrace{\frac{\partial \xi'}{\partial t}}_{\textbf{Defor-mation}}$$

The first step is to separate the particle velocity into two parts - one due to the general transport motion and the other due to material deformation.

$$\underbrace{\frac{\partial \xi_L}{\partial t}}_{\textbf{Total}} = \underbrace{V_i}_{\textbf{Transport motion}} + \underbrace{\frac{\partial \xi_E}{\partial t} + V_i \frac{\partial \xi_E}{\partial x}}_{\textbf{Defor-mation}}$$

The second step is to apply the Euler description to the deformation term. Subscripts are now used to identify the displacement variable as either Lagrange (L) or Euler (E).

The transformation equations (Cont.)

$$\underbrace{\frac{\partial \xi_L}{\partial x}}_{\textbf{Total}} = \underbrace{\varepsilon_1}_{\textbf{Initial Strain}} + \underbrace{\frac{\partial \xi_E}{\partial x}}_{\textbf{Variable}}$$

To be consistent, the strain will also be separated into two parts - a constant component that exists at time zero when the only particle velocity is the transport motion, plus a component due to the disturbance.

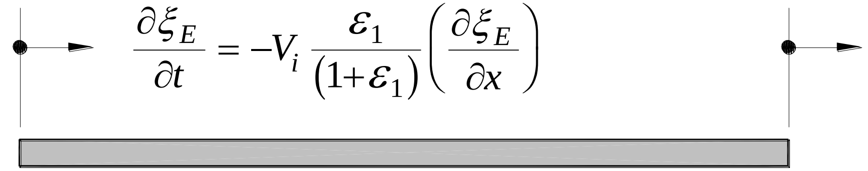
The Wave equation for a moving medium

Boundary Condition I

$$\frac{\partial \xi_E}{\partial t} = -V_i \frac{\varepsilon_1}{(1+\varepsilon_1)}\left(\frac{\partial \xi_E}{\partial x}\right)$$

Boundary Condition II

$$\frac{\partial \xi_E}{\partial t} = -V_i \frac{\partial \xi_E}{\partial x} + f(t)$$



$$\underbrace{\rho_o\left(\frac{\partial^2 \xi_E}{\partial t^2} + 2V_i \frac{\partial^2 \xi_E}{\partial x \partial t} + \frac{\partial^2 \xi_E}{\partial x^2} V_i^2\right)}_{\text{Inertia}} = \underbrace{E \frac{\partial^2 \xi_E}{\partial x^2}}_{\text{Elasticity}}$$

E.D. P.D.E.

Applying the Euler description to the one-dimensional wave equation and to the boundary conditions produces the model above. The P.D.E. is the wave equation for a moving medium. It appeared in one of the earliest traveling string papers. In that instance it applied to out-of-plane motion.

The solution

- Laplace transforms are used to solve the P.D.E.
- Although the wave equation for a moving medium appears often in the traveling string literature, the advantage of using the Laplace transform method on it seems to have been missed.
- It provides a closed form solution that can be easily modified for a variety of inputs.
- Refer to the paper for details.

The P.D.E. solution (Cont.)

- The solution for strain is:

$$\frac{\partial \xi_L}{\partial x} = \varepsilon_1 + \boldsymbol{L}^{\boldsymbol{-1}}\left\{ \frac{\boldsymbol{L}f(t)}{C}\left[e^{\frac{s(x-L)}{C-V_i}} + \beta e^{-\frac{sx}{C+V_i}-\frac{sL}{C-V_i}} \right]\sum_{n=0}^{m} \beta^n e^{-\frac{2nLCs}{C^2-V_i^2}} \right\}$$

$$\beta = \frac{-V_i + (\varepsilon_1 + 1)C}{V_i + (\varepsilon_1 + 1)C} \qquad C = \sqrt{\frac{E}{\rho}}$$

C, the velocity of sound, is typically much larger than V_i. It is 470 to 1,800 m/s for polymers and about 5,000 m/s for steel and aluminum. β is always smaller than 1.

Step input

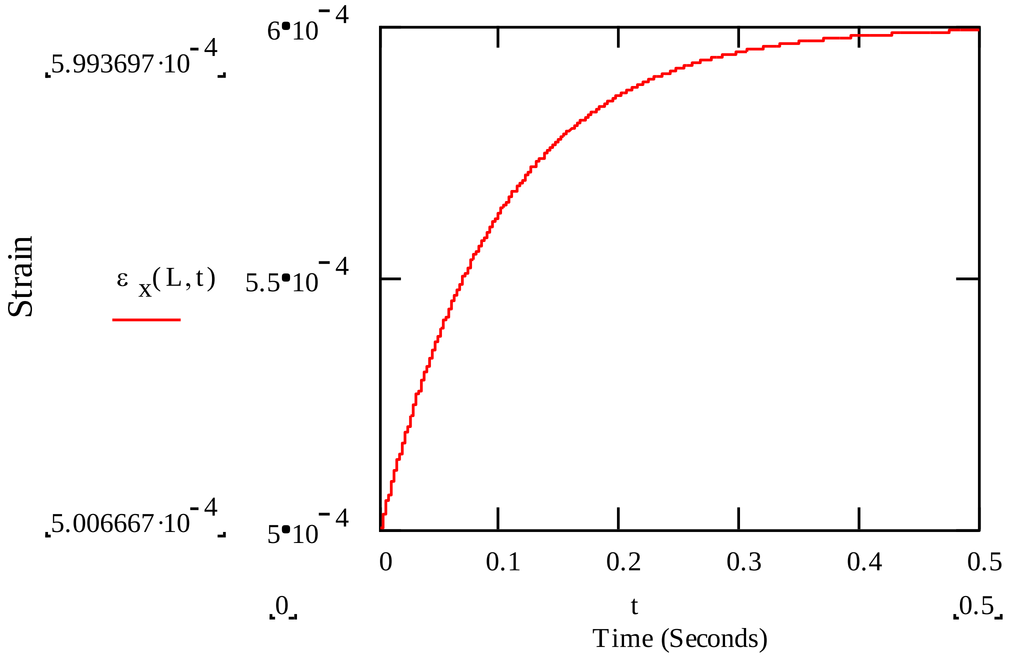
- Substituting the transform for a step input and inverting produces:

$$\frac{\partial \xi_L}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\left[\begin{array}{l}\beta^n \Phi\left(t - \frac{x}{C-V_i} - \frac{L}{C-V_i} - \frac{2nLC}{C^2 - V_i^2}\right) + \\ \beta^{n+1}\Phi\left(t - \frac{x}{C+V_i} - \frac{L}{C-V_i} - \frac{2nLC}{C^2 - V_i^2}\right)\end{array}\right]$$

$$m = t\,\frac{C^2 - V_i^2}{2LC} \qquad \beta = \frac{-V_i + (\varepsilon_1 + 1)C}{V_i + (\varepsilon_1 + 1)C}$$

Step response

- Strain at roller B for a .01% increase in B's speed.



Comparison with the O.D.E. model

- The model used for many years to analyze tension control problems can serve as a check for the P.D.E. At time scales where the propagation phenomena are imperceptible the models should agree. The O.D.E. model is:

$$L\frac{d\varepsilon_2(t)}{dt} = \left[V_{a0}\varepsilon_1 - V_{b0}\varepsilon_2(t)\right] + \left(V_b(t) - V_a(t)\right)$$

$$\varepsilon_1 = \frac{T_1}{A_0 E} \quad and \quad \varepsilon_2(t) = \frac{T_2(t)}{A_0 E}$$

O.D.E. comparison (Cont.)

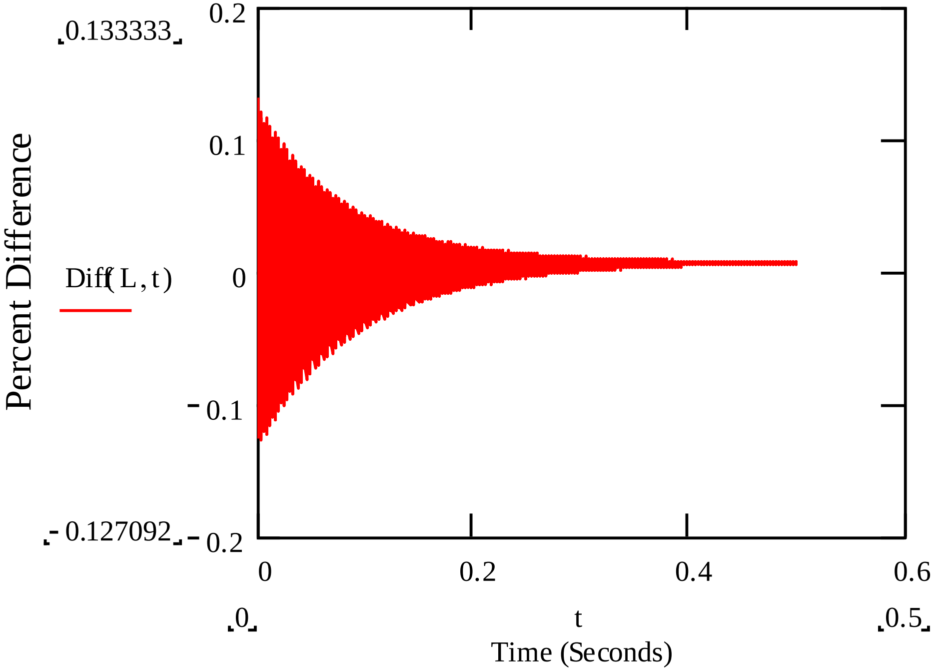
- The step response for the O.D.E. model is shown below. It can be shown, mathematically, that the P.D.E. converges to this when V_i << C. The P.D.E. solution is shown again for comparison.

$$\varepsilon_2(t) = \varepsilon_1 + \frac{\delta v}{V_i}\left(1 - e^{-\frac{V_i}{L}t}\right)$$

$$\frac{\partial \xi_L}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\left[\begin{array}{l}\beta^n \Phi\left(t - \frac{x}{C-V_i} - \frac{L}{C-V_i} - \frac{2nLC}{C^2 - V_i^2}\right) + \\ \beta^{n+1}\Phi\left(t - \frac{x}{C+V_i} - \frac{L}{C-V_i} - \frac{2nLC}{C^2 - V_i^2}\right)\end{array}\right]$$

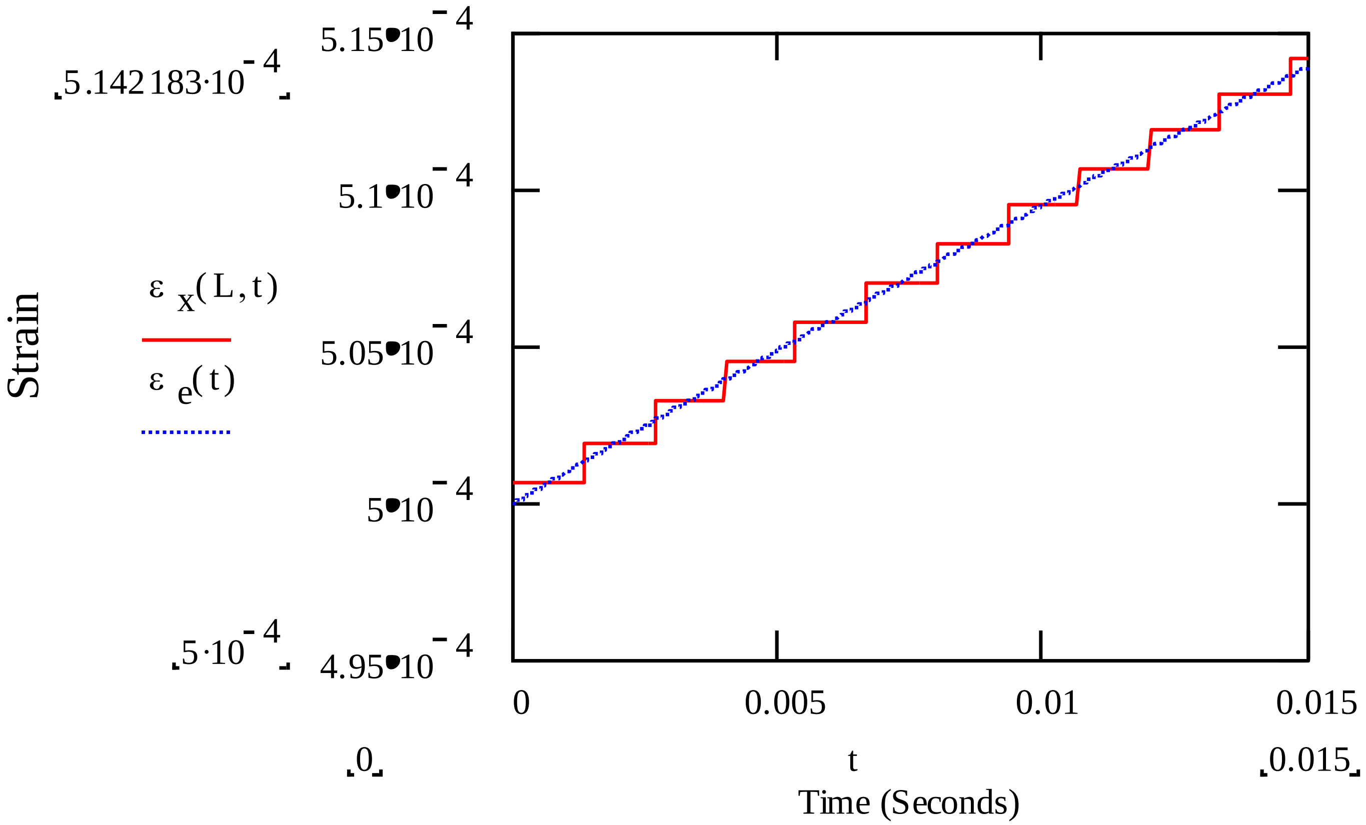
O.D.E comparison (Cont.)

- Comparison of step response of the O.D.E. and P.D.E. models for V_i = 10 m/s, C =1500 m/s, L = 1 m, ε_1= .0005, δv = .0001V_i



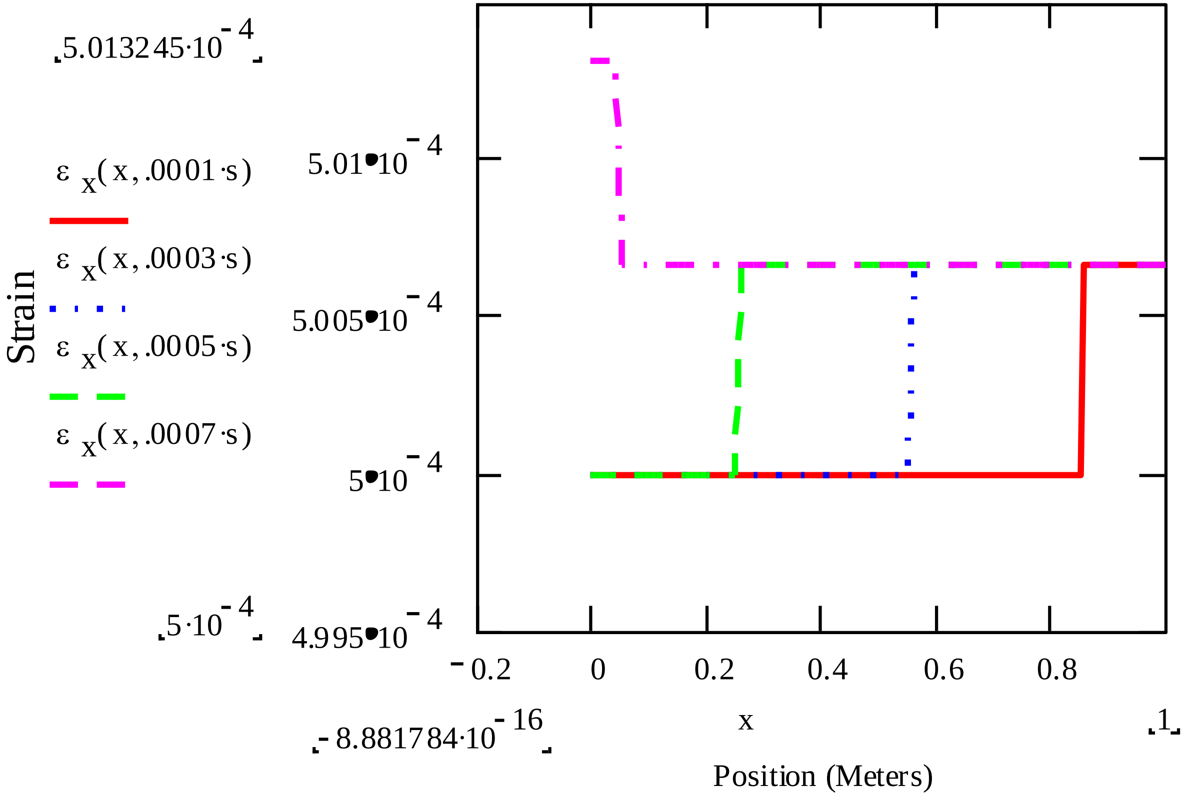
A high-resolution view of the solutions

- P.D.E. - solid red line. O.D.E. dashed blue line.



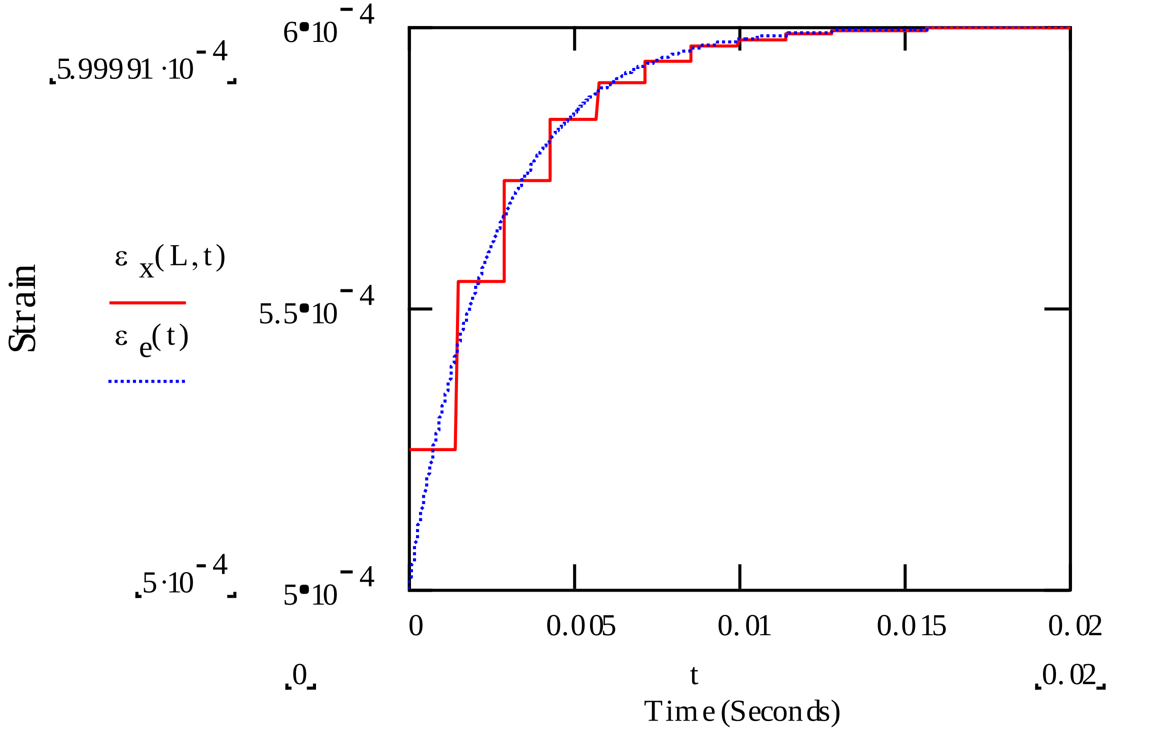
Propagation along the span

- The strain profile of the span is shown at 4 different times.



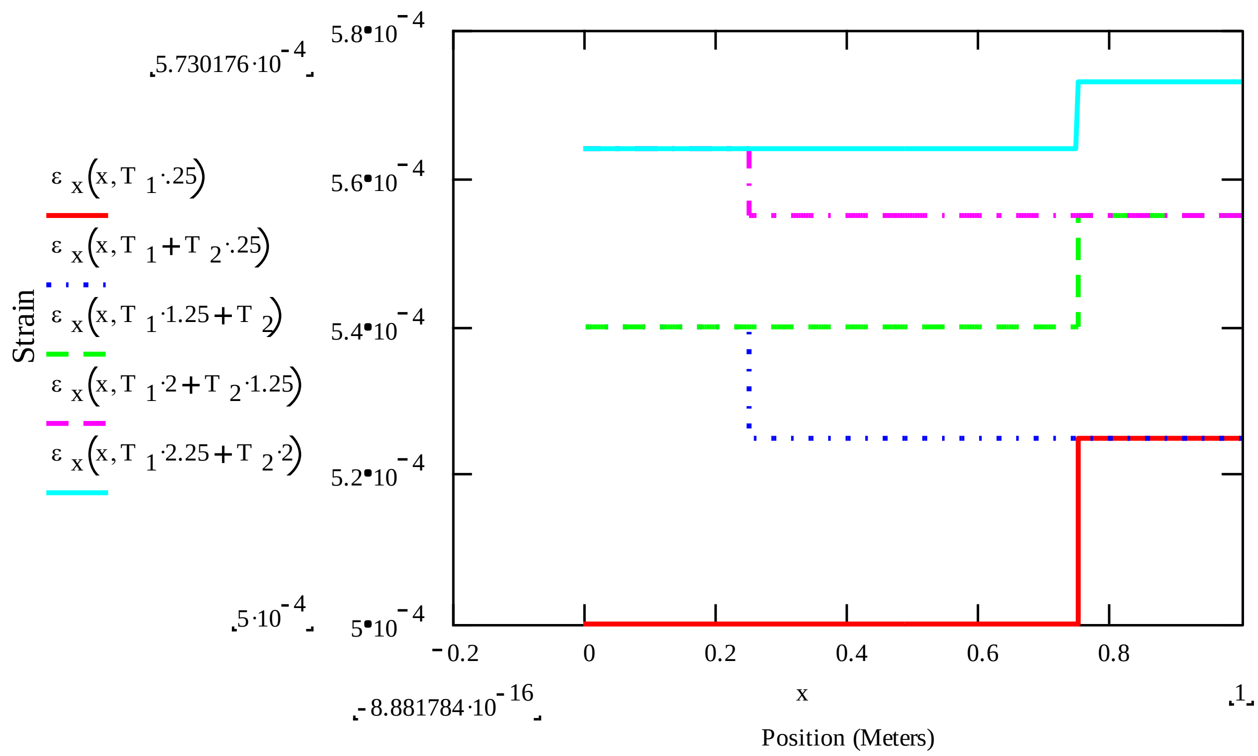
Step response at high speed

- Behavior of the P.D.E. is more obvious when the transport velocity gets close to C. For example, if V_i is 25% of C:



Propagation at high speed

- Here is the strain profile of the span at 5 different times for V_i = 25% of C.



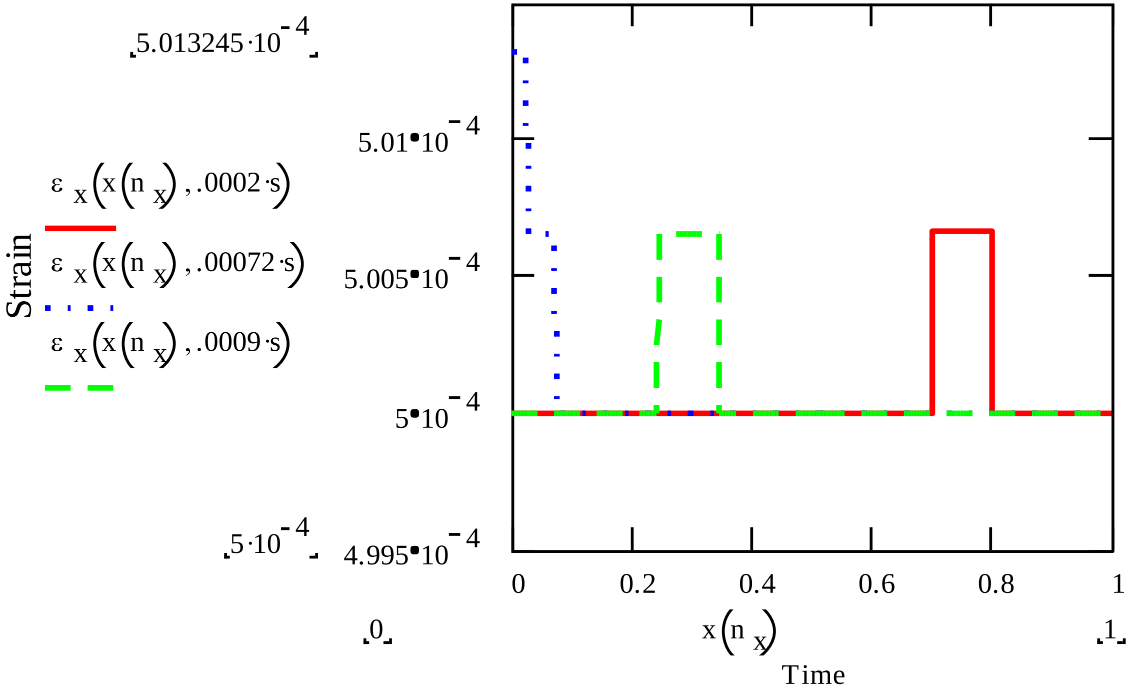
Response to a single pulse

- The solution for a single, brief pulse at roller B looks like this:

$$\frac{\partial \xi}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\left[\begin{array}{l}\beta^n\left[\begin{array}{l}\Phi\left(t+\frac{x}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)- \\ \Phi\left(t+\frac{x}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}-t_1\right)\end{array}\right] \\ +\beta^{n+1}\left[\begin{array}{l}\Phi\left(t-\frac{x}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)- \\ \Phi\left(t-\frac{x}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}-t_1\right)\end{array}\right]\end{array}\right]$$

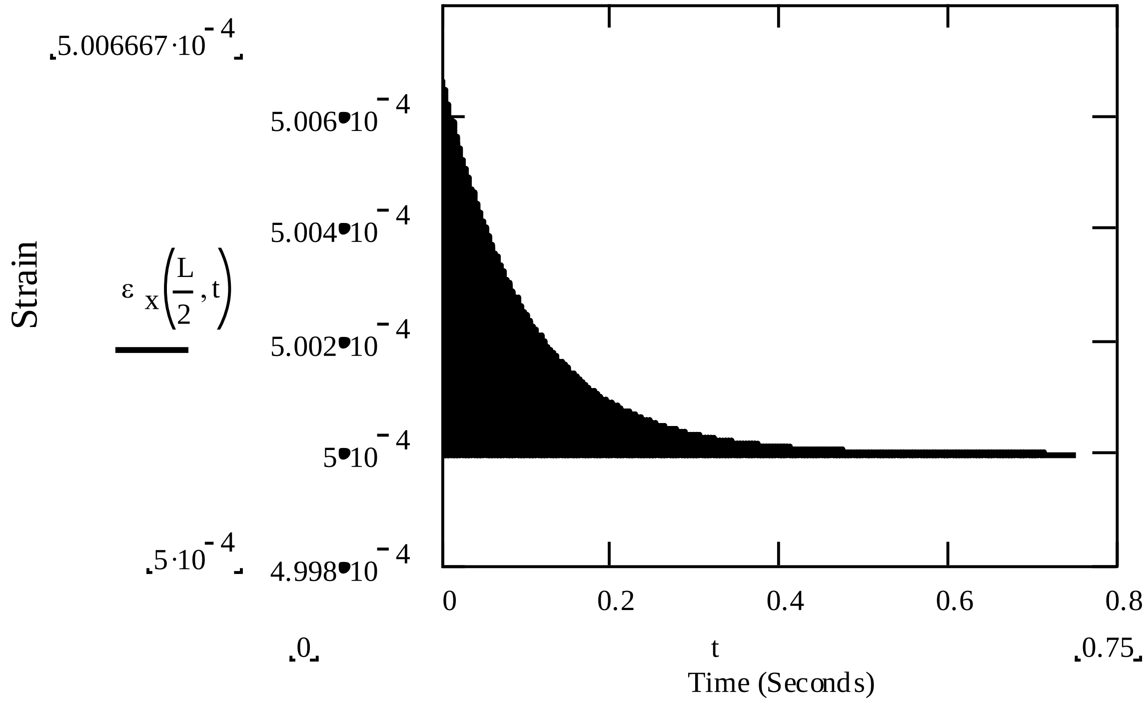
Response to a single pulse (Cont.)

- The graph below shows a pulse at three different times as it moves through the span.



Transport damping

- Decay of a single pulse: L = 1 m, V_i = 10 m/s, L/V_i = .1 s, C = 1500 m/s, V_i = .7 % of C

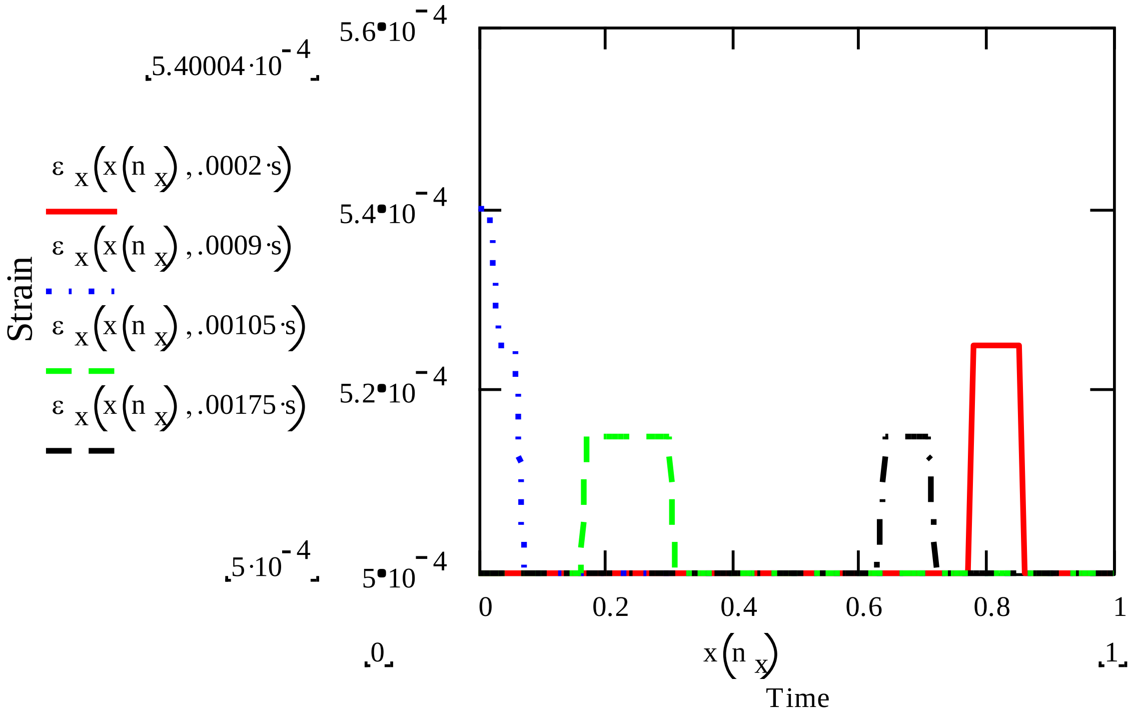


Energy transfers

- Miranker in 1960 pointed out that energy is not conserved in transverse oscillations of a moving string.
- It is apparent that similar conclusions apply to longitudinal strain variations.
- Energy can be exchanged with the rollers in the following way. When a pulse arrives at a roller there is a change in roller torque due to the tension change. And, since the roller is rotating, work is done, either on the roller or on the web.
- In addition, strain energy can be transferred to the downstream span during the time the pulse is at roller B.

A high-speed view

- If V_i is increased to 25% of C more detail becomes visible.



Repetitive pulses

- The amplitude for a single pulse is very small ($\delta v/C$). That may not be the case for a repetitive pulse.
- If the period is an integer fraction of the time for the pulse to travel up the span and back it will be reinforced and amplified.
- Depending on how much the pulse is damped by viscoelasticity and friction, it may even be amplified when the period is an integer multiple of the round-trip time.
- Without damping, the pulse can grow to the same amplitude as if it were a step change.

Sinusoidal disturbances

- The sinusoidal solution looks like this:

$$\frac{\partial \xi_L}{\partial x}(x,t) = \varepsilon_1 + \frac{\delta v}{C}\sum_{n=0}^{m}\beta^n \left[\begin{array}{l} \sin\left[\omega\left(t+\frac{x}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)\right]\bullet \\ \Phi\left(t+\frac{x}{C-V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)+ \\ \beta\sin\left[\omega\left(t-\frac{x}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right)\right]\bullet \\ \Phi\left(t-\frac{x}{C+V_i}-\frac{L}{C-V_i}-\frac{2nLC}{C^2-V_i^{\,2}}\right) \end{array}\right]$$

Sinusoidal disturbances (Cont.)

- A sinusoid will reinforce itself in the same manner as a repetitive pulse. It grows to an amplitude of $\delta v/V_i$, with a time constant of L/V_i.

$$\omega = 2\pi n \frac{\left(C^2 - V_o{}^2\right)}{2LC} \qquad n = 1, 2, 3 \cdots$$

$$\frac{\left(C^2 - V_o{}^2\right)}{2LC} = \frac{1}{L/\left(C - V_i\right) + L/\left(C + V_i\right)}$$

High speed behavior

- The condition for amplification of a sinusoidal disturbance raises interesting questions. As the transport speed approaches C, the resonant frequencies become zero for all n. Also, the upstream velocity becomes zero. Could one see a standing wave of zero frequency?

$$\omega = 2\pi n \frac{\left(C^2 - V_o{}^2\right)}{2LC} \qquad n = 1, 2, 3 \cdots$$

High speed behavior (Cont.)

- Study of the traveling string literature suggests that a more sophisticated model is needed when the transport velocity is a significant fraction of the speed of sound, C.
- In particular, the variation in mass per unit length (caused by strain) should be included in the model. This creates a nonlinear problem requiring numerical methods.

Conclusions

- This model has two principal uses.
  - As a step toward more realistic models. These will likely require numerical methods. Having a closed-form solution for simple cases will provide a way to check the programs.
  - As a diagnostic tool for problems whose causes may have been unrecognized in the past.
- Some shortcomings that could be addressed in more sophisticated models are:
  - There is no provision for variation in mass per unit length. The variation is small but it could be important because conservation of mass is central to tension behavior in webs.

Conclusions (Cont.)

  - There is no provision for viscoelasticity, damping or friction. Viscoelasticity will have a particularly strong effect on the velocity (dispersion) and amplitude of disturbances.
  - Laboratory work should be done before going on to more complex models.
- The following insights may help in understanding current problems on process lines.
  - If a strain pulse is brief (of the order of L/C) it will be very small. The amplitude compared to a continuous step will be smaller by V_i/C.
  - The upstream velocity of propagation is $C-V_i$. The downstream velocity is $C+V_i$.

Conclusions (Cont.)

- When the speed of a roller in a span increases suddenly to a new steady value, the initial change in strain will be on the order of $\delta v/C$. The total strain in the span grows through a process of propagation, reflection and accumulation. On large time scales the propagation behavior is imperceptible and the results conform to the O.D.E. model.
- A single brief pulse will be damped as it is reflected back and forth between the rollers of the span. This transport damping will contribute to other energy losses that remove energy from the pulse.

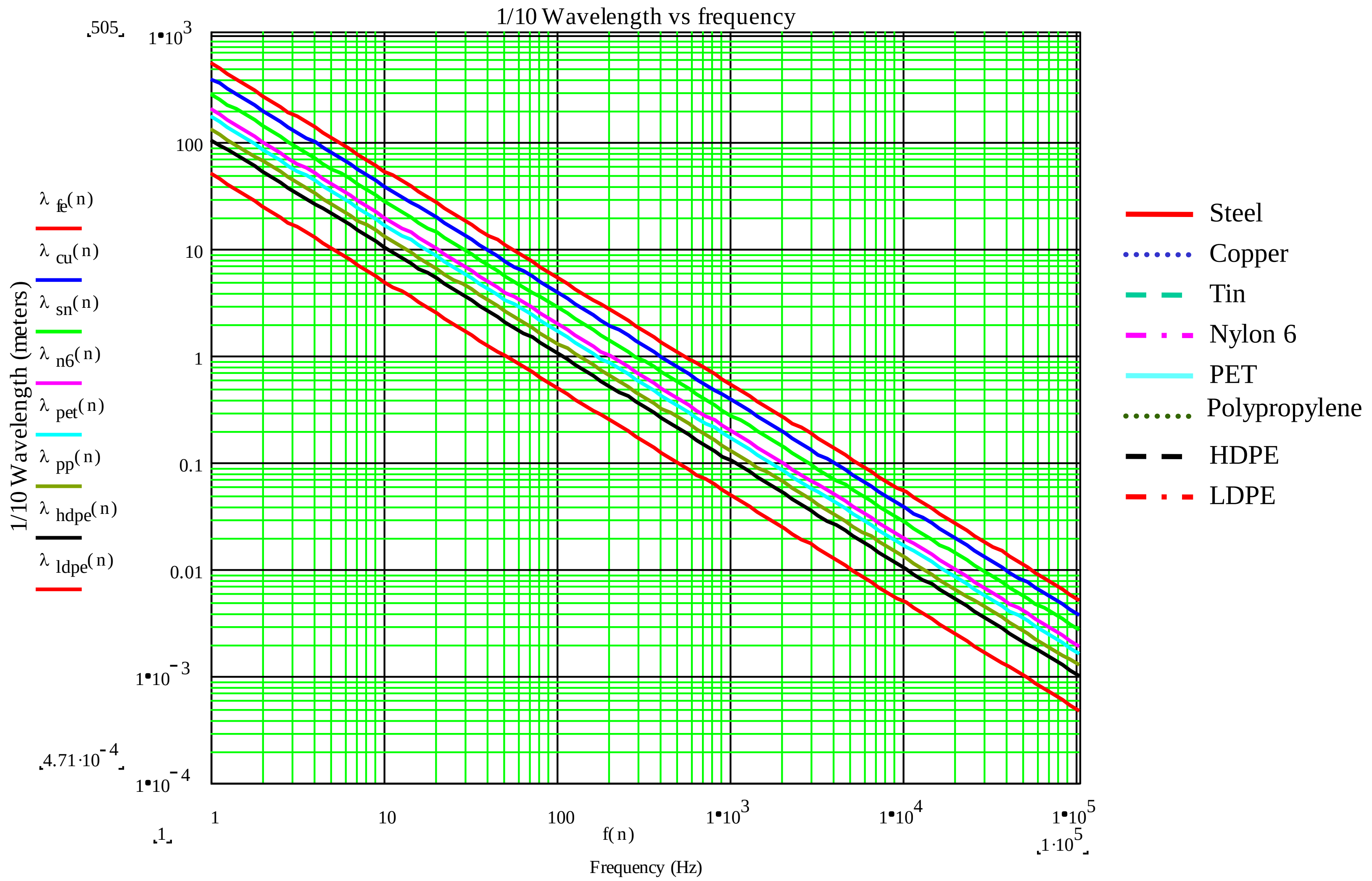
Conclusions (Cont.)

- An example of a single pulse is a sudden slip on a roller due to a splice.
- A repetitive disturbance can be amplified if its period is an integer fraction (or integer multiple if damping is low) of the time for the pulse to travel upstream and back. The final amplitude can be as large as if the pulse were a step.
- An example of a repetitive pulse would be the cyclic disturbance of an embossing roller.

When is a web slender?

- That depends on the frequency spectrum of the disturbance. If the width of the web is less than 1/10 of the wavelength of the highest frequency component, then transverse effects will be negligible.
- A graph of wavelength versus frequency for several different materials is shown on the next visual.

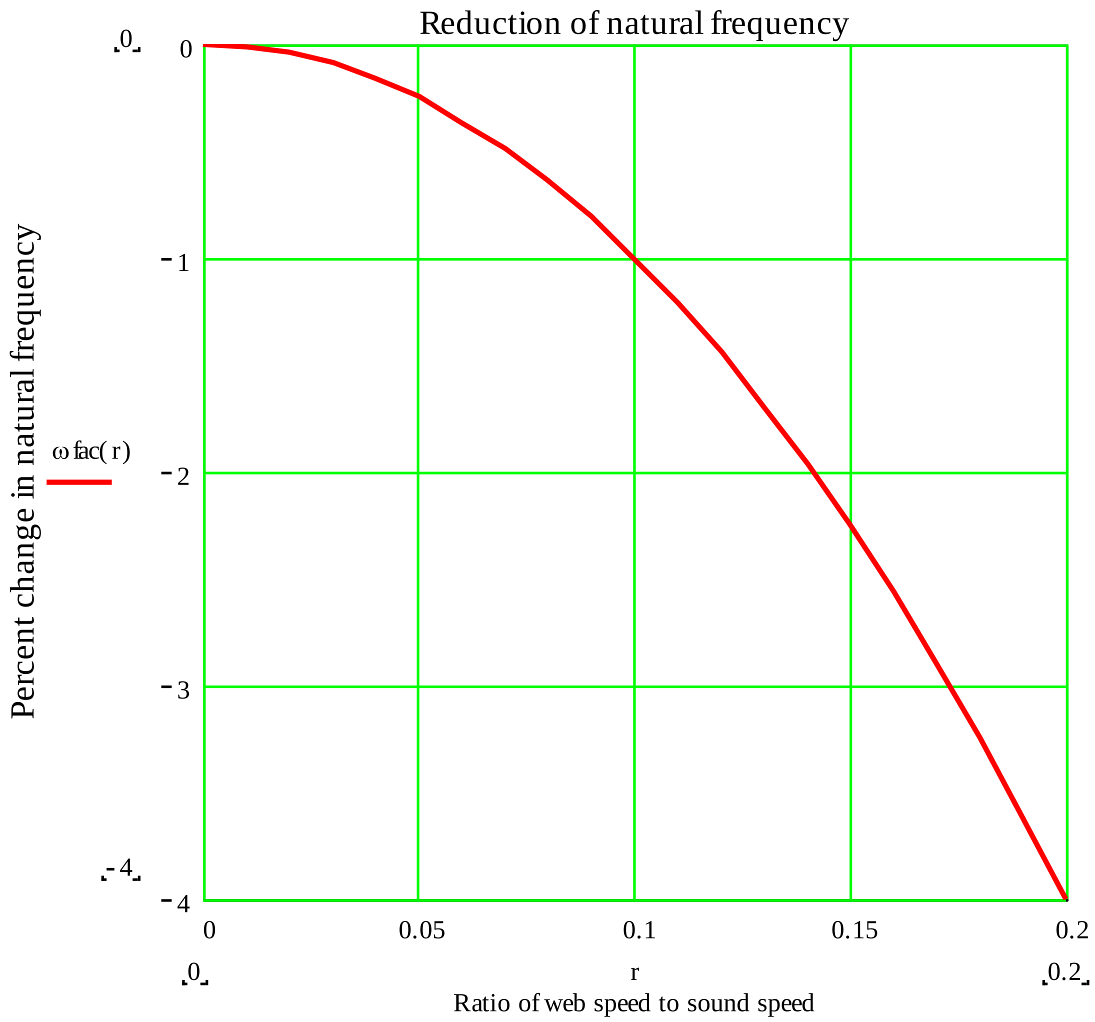
$$\frac{\lambda}{10} = \frac{C}{10f}$$


1/10 Wavelength vs frequency
1/10 Wavelength (meters)
Frequency (Hz)
f(n)
λ fe(n)
λ cu(n)
λ sn(n)
λ n6(n)
λ pet(n)
λ pp(n)
λ hdpe(n)
λ ldpe(n)
Steel
Copper
Tin
Nylon 6
PET
Polypropylene
HDPE
LDPE
505
1•10^3
100
10
1
0.1
0.01
1•10^-3
4.71·10^-4
1•10^-4
1
10
100
1•10^3
1•10^4
1•10^5
.1
1·10^5

How much does web speed affect longitudinal natural frequencies?

- At realistic web speeds the resonant frequencies are affected very little.
- The ratio of moving to stationary resonant frequencies is shown below. This function is plotted on the next visual.

$$\frac{f_{n\,(moving)}}{f_{n\,(stationary)}} = 1 - \left(\frac{V_i}{C}\right)^2$$

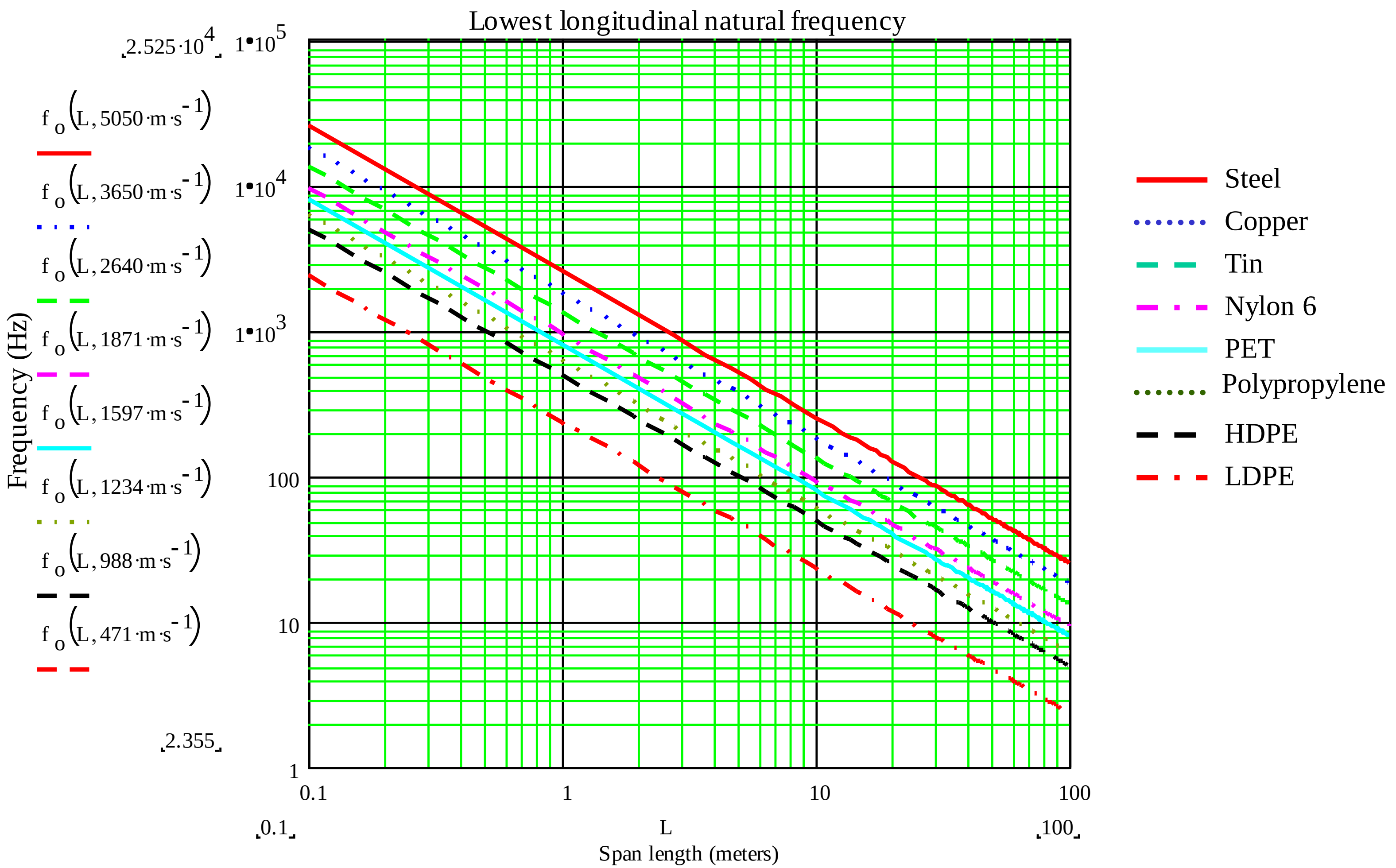


Reduction of natural frequency
Percent change in natural frequency
ω fac(r)
0
- 1
- 2
- 3
- 4
0
- 4
0
0.05
0.1
0.15
0.2
0
0.2
r
Ratio of web speed to sound speed

What are typical values for longitudinal resonant frequencies?

- The next visual shows typical resonant frequencies (n=1) for eight different materials over a range of span lengths.
- The frequencies are for zero web speed. For a moving web the correction factor in the previous graph should be applied.

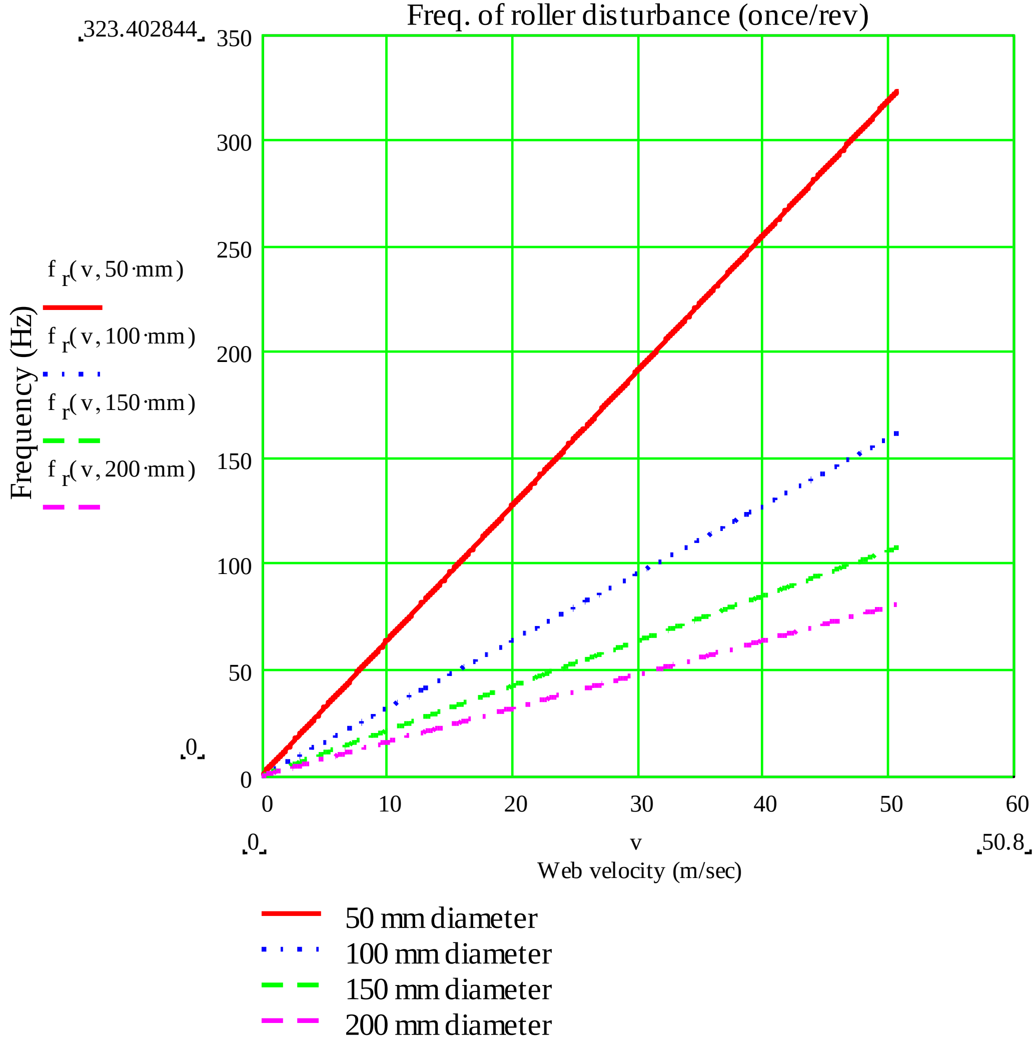
$$f_{n=1(fixed)} = \frac{C}{2L}$$



Lowest longitudinal natural frequency
Frequency (Hz)
Span length (meters)
L
Steel
Copper
Tin
Nylon 6
PET
Polypropylene
HDPE
LDPE
f o(L, 5050·m·s-1)
f o(L, 3650·m·s-1)
f o(L, 2640·m·s-1)
f o(L, 1871·m·s-1)
f o(L, 1597·m·s-1)
f o(L, 1234·m·s-1)
f o(L, 988·m·s-1)
f o(L, 471·m·s-1)
2.525·10^4
2.355
1•10^5
1•10^4
1•10^3
100
10
1
0.1
1
10
100
0.1
100

What frequencies can be expected for disturbances?

- Besides the obvious case of an eccentric roller:
  - Printing could produce a small disturbance at each repetition of the repeat length.
  - An embossing cylinder could produce a disturbance for each repeat of the pattern.
  - A vacuum roller, used to increase traction, could produce a disturbance as each row of holes in the roller releases the web.
- The next visual shows the frequency of one disturbance per revolution of a roller as a function of web speed and roller diameter.
- The frequency from this graph should be multiplied by the number of repeats per revolution.


Freq. of roller disturbance (once/rev)
323.402844
350
300
250
200
150
100
50
0
0
Frequency (Hz)
f r(v, 50·mm)
f r(v, 100·mm)
f r(v, 150·mm)
f r(v, 200·mm)
0
10
20
30
40
50
60
0
v
50.8
Web velocity (m/sec)
50 mm diameter
100 mm diameter
150 mm diameter
200 mm diameter

$$f = \frac{V_i}{\pi D}$$