The Use of Conservation of Mass in Modeling Lateral Behavior in Moving Webs

By

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Three main topics will be covered

- An improved conceptual framework for solving lateral web handling problems - an extension of work presented at IWEB 2005. It is based on,
  - The principle of conservation of mass
  - Nonlinear elasticity theory.
- An efficient method for using these ideas by treating all webs, flat or otherwise, as membranes in a two-dimensional frame of reference.
- An explanation of how all of the these ideas are applied to the problem of modeling a baggy web.

A pitch for elasticity theory in web handling

- It provides a natural framework for incorporating the lateral effects of conservation of mass – which the others don't.
- It's the foundation for all the other methods we use.
- It's now possible to solve the equations with desktop computer tools.
- These tools are fast enough to be used as learning aids for elasticity theory.
- If you notice a tutorial quality about this presentation, that's because I'd like to help other people to pursue this approach.

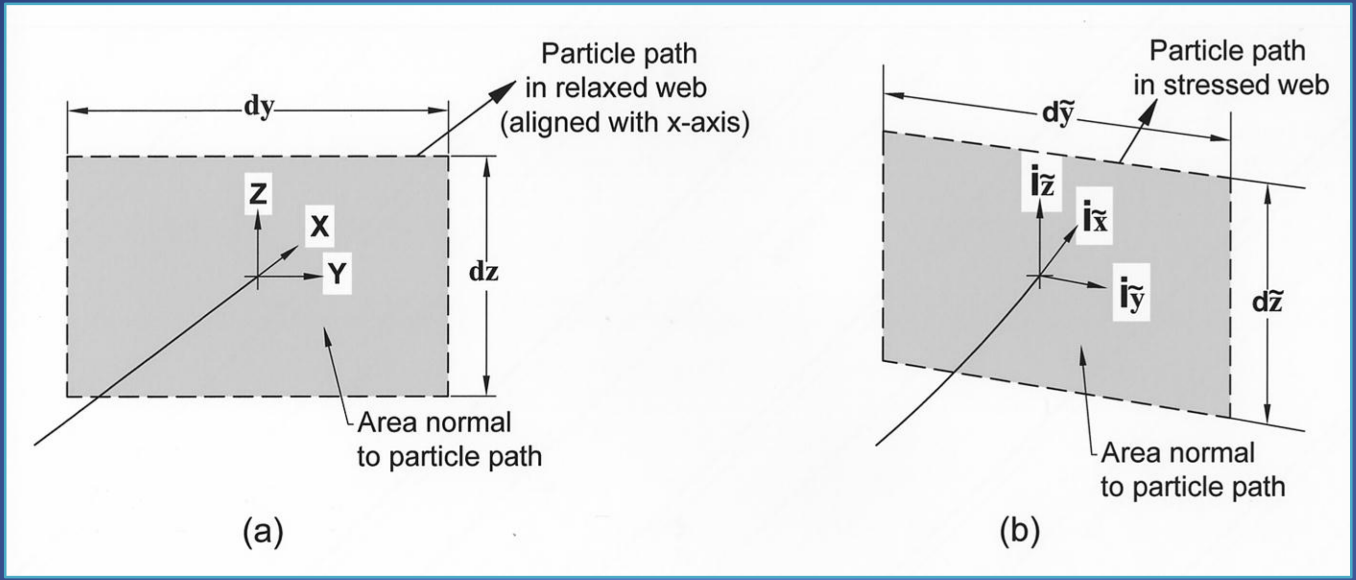
The history of $\frac{V_2}{V_1} = \frac{1+\varepsilon_2}{1+\varepsilon_1}$ in webs

- Osborne Reynolds first used it to explain creep in belts on pulleys in 1874.
- Shelton applied it in a 1986 paper on tension control and called it "concept of transport of strain".
- The author used it in a two-dimensional sense as the basis for one of the boundary conditions in a 2005 IWEB paper and called it the "normal strain rule".
- It's actually a corollary to a more fundamental concept – The Velocity-Strain Equation

Nonlinear elasticity theory

- The type discussed here deals with problems like this – small strains, but large rotations. This obviously includes other kinds of thin curved shapes.
- For web handling it's necessary not only for large rotation problems like twisted or baggy webs. It's also needed to deal with the interaction of MD stress with small rotations.
- In 1953 Novozhilov published a wonderful monograph on the subject in which he derives and explains the equations without the use of tensors and then shows how they can be simplified for small strain. And, most importantly, he explains the physical interpretation and limits of application of everything as he goes.
- For web handling, it's essential. Linear theory is practically useless for modeling lateral web behavior.

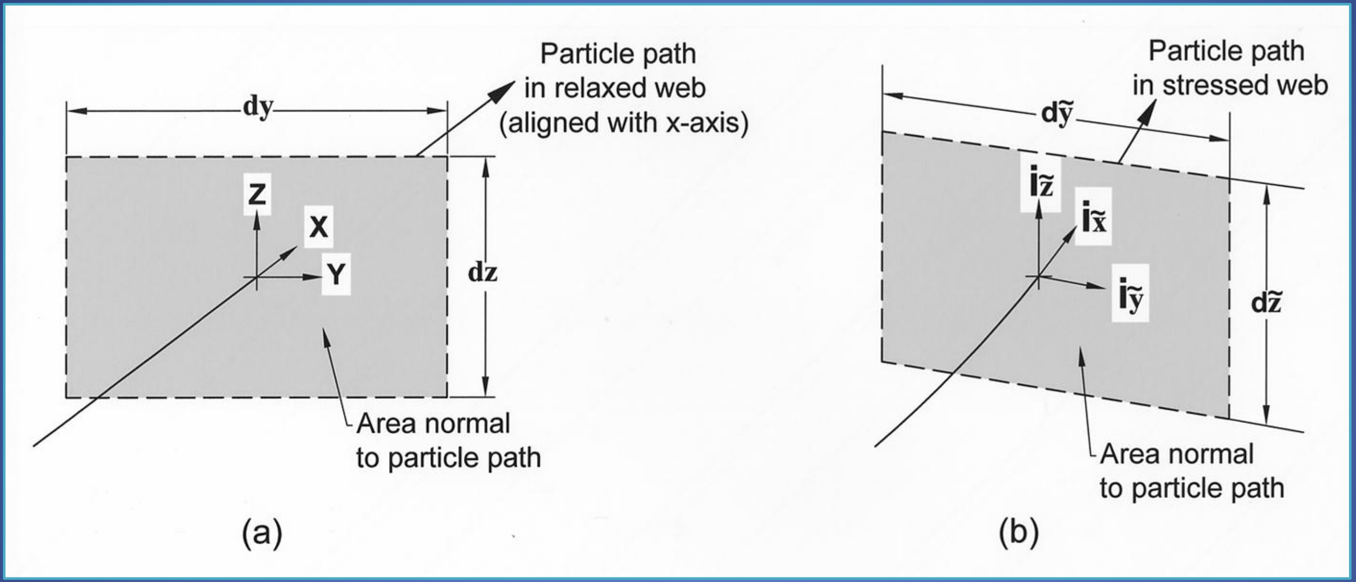
Requirements for comparing mass flows



Three things needed for calculating mass flow:

- Paths followed by the web particles
- Cross sectional area
- Density of the material

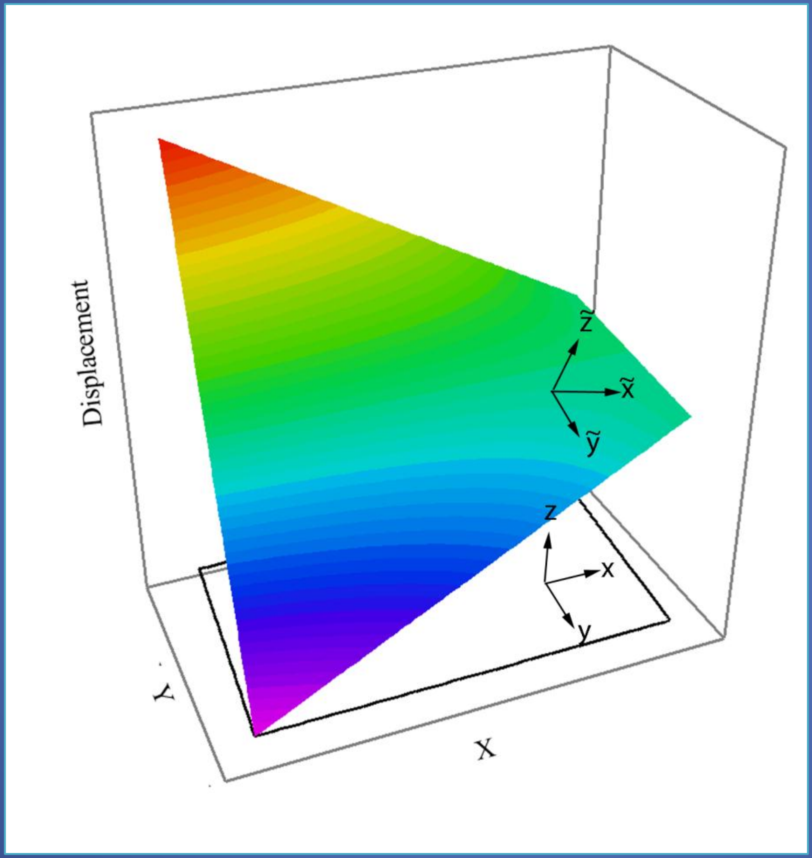
Nonlinear elasticity is the key



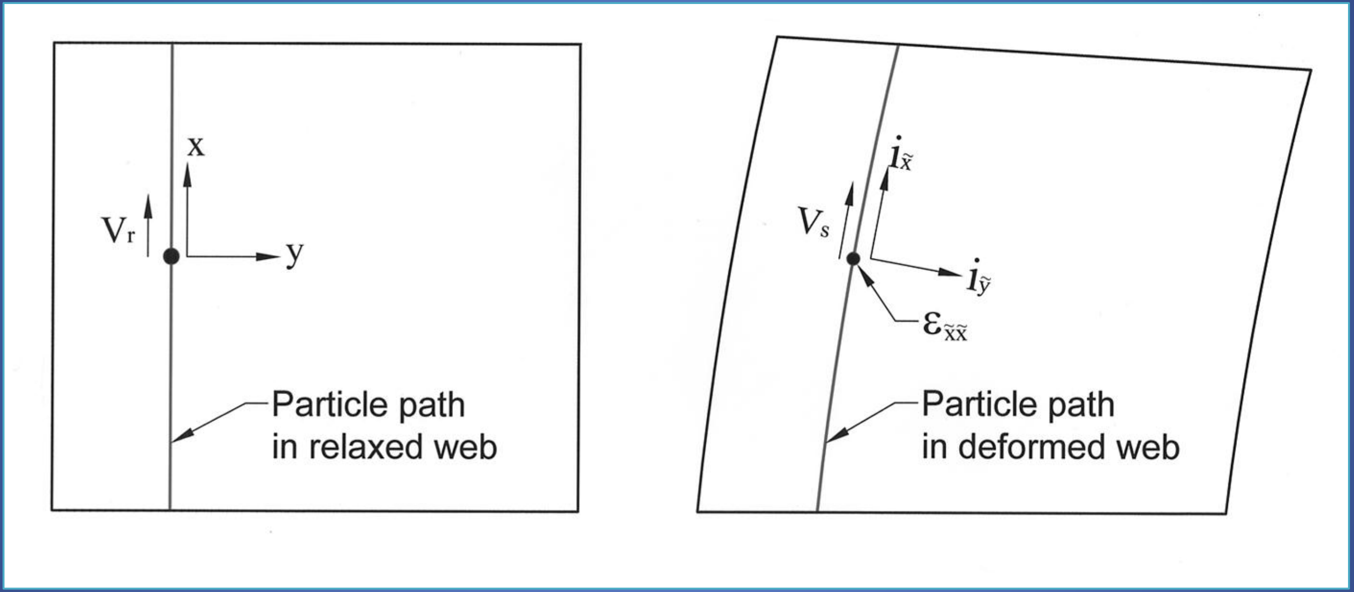
It provides:

- The direction cosines for $\tilde{x}, \tilde{y}, \& \tilde{z}$ relative to x, y and z
- Cross sectional area $d\tilde{z} = dz\left(1+\varepsilon_{\tilde{z}\tilde{z}}\right) \quad d\tilde{y} = dy\left(1+\varepsilon_{\tilde{y}\tilde{y}}\right)$
- Density of the material $\rho_s = \dfrac{\rho_r}{\left(1+\varepsilon_{\tilde{x}\tilde{x}}\right)\left(1+\varepsilon_{\tilde{y}\tilde{y}}\right)\left(1+\varepsilon_{\tilde{z}\tilde{z}}\right)}$

Deformed coordinates on twisted web



The Velocity-Strain Equation



$$V_r = \frac{V_s}{\left(1+\varepsilon_{\tilde{x}\tilde{x}}\right)} \quad \text{and} \quad V_{ravg} = \frac{V_{savg}}{1+\varepsilon_{\tilde{x}\tilde{x}avg}}$$

Ok, so what's the big deal

It's this. The MD strain boundary condition at the downstream roller can now be defined in terms of only the conditions at that roller. There is no need to know anything about what's happening at the upstream roller. And this is critical for non-uniform web models like a baggy web.

Normal strain rule $$\frac{V_{downstream}}{V_{upstream}} = \frac{1+\varepsilon_{\tilde{x}\tilde{x}downstream}}{1+\varepsilon_{\tilde{x}\tilde{x}upstream}}$$

With the normal strain rule it was necessary to know the strain at the entry of the upstream roller. That was ok for a misaligned or concave roller, because uniform stress could be assumed in the upstream span. But, you can't do that for a non-uniform web. However, with the velocity-strain equation, *Vs* becomes the surface velocity of the roller and the ratio of *Vs* to *Vr* can be calculated from the target values for the averages plus information about the shape of the roller and/or the web .

Velocity-strain Eq. $$V_r = \frac{V_s}{\left(1+\varepsilon_{\tilde{x}\tilde{x}}\right)} \quad \text{and} \quad V_{ravg} = \frac{V_{savg}}{1+\varepsilon_{\tilde{x}\tilde{x}avg}}$$

Notation definitions

Displacements in *x*, *y* and *z* directions are *u*, *v* and *w*, respectively

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{zz} = \frac{\partial w}{\partial z}$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad e_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Equations of equilibrium for the 2D+*w* model

(No terms involving z)

Projection of Cauchy stresses on the x-axis

Cauchy stresses - aligned with the $\tilde{x}$ and $\tilde{y}$ axes

$$\frac{\partial}{\partial x}\left[\left(1+e_{xx}\right)\sigma_{\tilde{x}\tilde{x}}+\left(\frac{1}{2}e_{xy}-\omega_z\right)\sigma_{\tilde{x}\tilde{y}}\right]+\frac{\partial}{\partial y}\left[\left(1+e_{xx}\right)\sigma_{\tilde{x}\tilde{y}}+\left(\frac{1}{2}e_{xy}-\omega_z\right)\sigma_{\tilde{y}\tilde{y}}\right]=0$$

x-direction Eq.

Cosine of angle between $\tilde{x}$ and x

Cosine of angle between $\tilde{y}$ and x

$$\frac{\partial}{\partial x}\left[\left(\frac{1}{2}e_{xy}+\omega_z\right)\sigma_{\tilde{x}\tilde{x}}+\left(1+e_{yy}\right)\sigma_{\tilde{x}\tilde{y}}\right]+\frac{\partial}{\partial y}\left[\left(\frac{1}{2}e_{xy}+\omega_z\right)\sigma_{\tilde{x}\tilde{y}}+\left(1+e_{yy}\right)\sigma_{\tilde{y}\tilde{y}}\right]=0$$

y-direction Eq.

$$\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}\sigma_{\tilde{x}\tilde{x}}+\frac{\partial w}{\partial y}\sigma_{\tilde{x}\tilde{y}}\right]+\frac{\partial}{\partial y}\left[\frac{\partial w}{\partial x}\sigma_{\tilde{x}\tilde{y}}+\frac{\partial w}{\partial y}\sigma_{\tilde{y}\tilde{y}}\right]=0$$

z-direction Eq.

Nonlinear theory of shells for membranes??

- The 2D+*w* equations, when combined with curvilinear coordinates, constitute a nonlinear theory of shells for membranes.
- When viewed from the perspective of the literature on this subject, they might be considered naïve and treacherous – unstable solutions, only certain types of boundary conditions allowed, etc.
- But for web handling, there is a simple way to make them safe and useful.

Strain in the $\tilde{x}, \tilde{y}$ coordinate system

$$\tilde{x} - direction \qquad \varepsilon_{\tilde{x}\tilde{x}} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\tilde{y} - direction \qquad \varepsilon_{\tilde{y}\tilde{y}} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$shear\ in\ \tilde{x}–\tilde{y}\ plane \qquad \varepsilon_{\tilde{x}\tilde{y}} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

See the paper for an example of how these work.

Stress-strain relations (Hook's law)

$$\sigma_{\tilde{x}\tilde{x}} = \frac{E}{1-\mu^2}\left(\varepsilon_{\tilde{x}\tilde{x}} + \mu\varepsilon_{\tilde{y}\tilde{y}}\right)$$

$$\sigma_{\tilde{y}\tilde{y}} = \frac{E}{1-\mu^2}\left(\varepsilon_{\tilde{y}\tilde{y}} + \mu\varepsilon_{\tilde{x}\tilde{x}}\right)$$

$$\sigma_{\tilde{x}\tilde{y}} = \frac{E}{2\left(1+\mu\right)}\left(\varepsilon_{\tilde{x}\tilde{y}}\right)$$

Normal entry condition

Cosine of angle between $\tilde{y} = x$

Cosine of angle between $\tilde{x}$ *and* x

Unit vector $i_{\tilde{x}}$ in terms of i and j of the *x*-*y* coordinate system.

$$i_{\tilde{x}} = i\,\frac{1+e_{xx}}{1+E_x} + j\,\frac{\frac{1}{2}e_{xy}+\omega_z}{1+E_x}$$

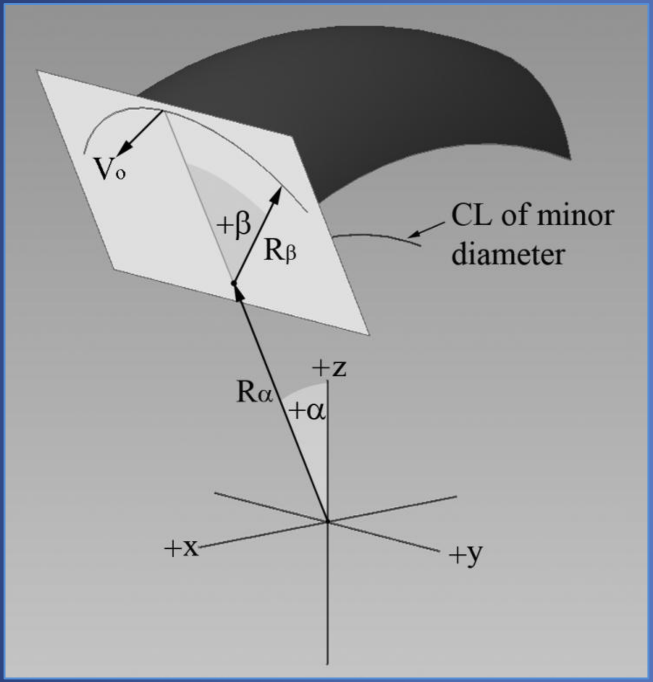
So, the tangent of angle ψ between $\tilde{x}$ *and* x is:

$$\tan(\psi) = \frac{\frac{1}{2}e_{xy}+\omega_z}{1+e_{xx}} = \frac{\frac{\partial v}{\partial x}}{1+e_{xx}}$$

For a flat, rectangular web

Ψ is set equal to the angle of the down-stream roller axis, relative to the *y*-axis.

The baggy web



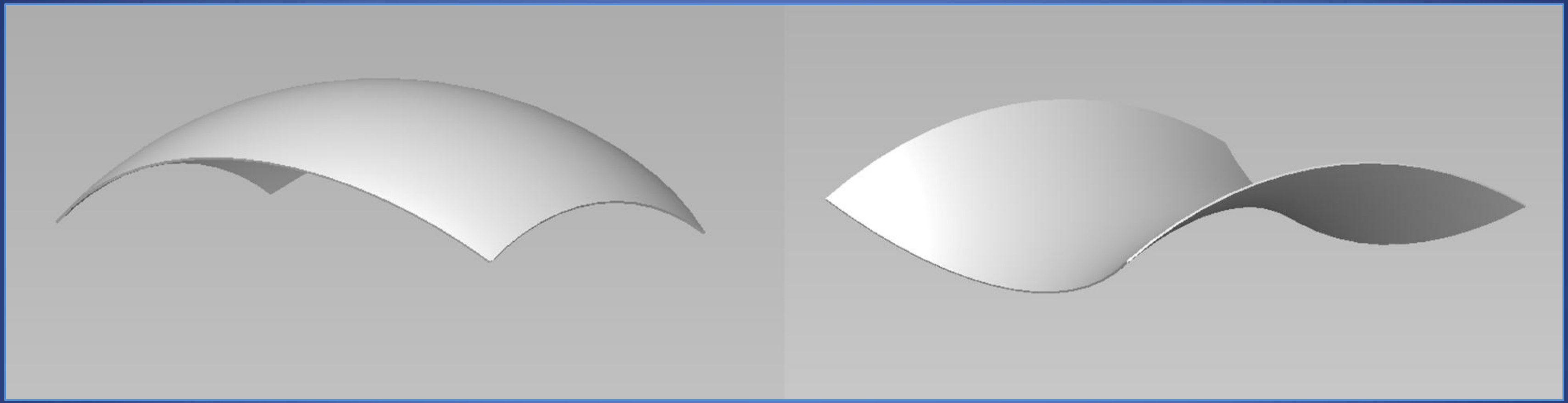
$$z = \left(R_\alpha + R_\beta \cos(\beta)\right)\cos(\alpha)$$

$$x = \left(R_\alpha + R_\beta \cos(\beta)\right)\sin(\alpha)$$

$$y = R_\beta \sin(\beta)$$

- Elastic analysis requires we start with a relaxed reference shape.
- It’s got to allow web motion without creating strains. That suggests a surface of revolution.
- When you consider the causes of bagginess, it’s natural to think of a segment of a torus-like shape.
- So, we’ll start with a simple torus.

Three basic types of surface curvature



Elliptic
Centers of the principal curves are on same side.
All points are elliptic on this one.
(Outer part of a torus)

Hyperbolic
Centers of principal curves are on opposite sides.
All points are hyperbolic on this one.
(Inner part of a torus)

Parabolic
Curvature of zero in one direction
Example: All points on a cylinder

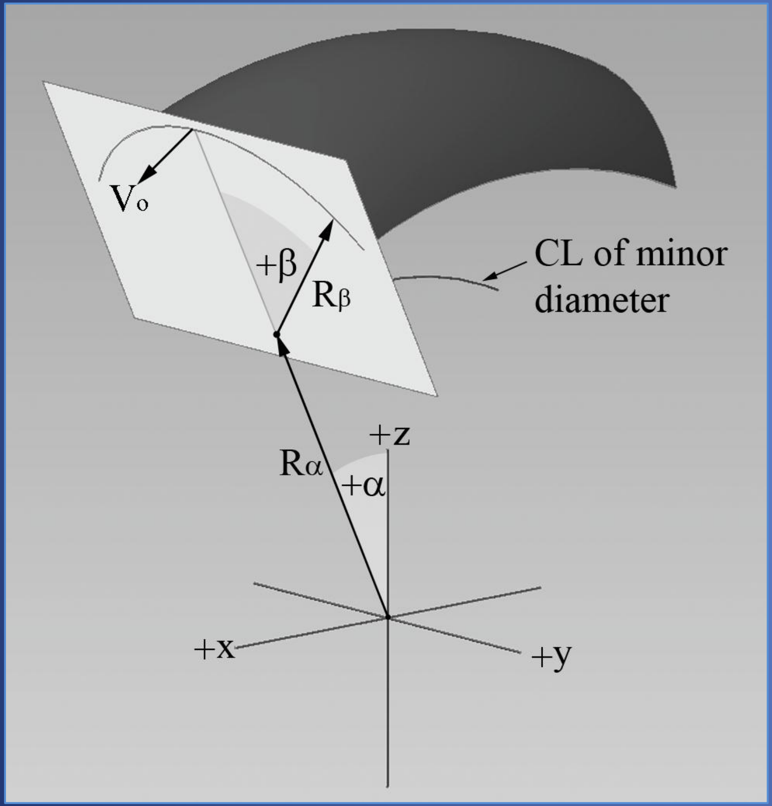
Curvilinear coordinates

- To use the 2D+*w* model, the toroidal surface must be "flattened".
- This is done with curvilinear coordinates in which α will replace the *x*-coordinate and β will replace *y*.
- The displacements will continue to be labeled as *u*, *v* and *w*. They will still have units of distance.
- The displacement *w* will be defined as being along a normal to the relaxed surface with $w = 0$ being at the surface itself.

The 2D+*w* model in curvilinear coordinates

- There is nothing out of the ordinary in the conversion of the 2D+*w* model to toroidal curvilinear coordinates and the methods are covered in numerous references. So, there is no reason to take the time to discuss them here.
- The final equations are presented in the paper using coordinates α and β. Tildes are once again used to indicate the coordinates after they have been deformed by stress.
- Furthermore, all of the relationships between the relaxed and deformed coordinates that held for the Cartesian system also hold for the curvilinear α-β coordinates.

Applying the velocity-strain equation



Velocity of the surface of the torus as a function of *Vo* and β is,

$$V_r = V_o \frac{R_\alpha + R_\beta \cos(\beta)}{R_\alpha + R_\beta}$$

The velocity-strain equation provides MD strain at the downstream boundary as a function of *Vs*, *Vo* and β.

$$\varepsilon_{\tilde{\alpha}\tilde{\alpha}bndry} = \frac{V_s}{V_o} \frac{\left(R_\alpha + R_\beta\right)}{\left(R_\alpha + R_\beta \cos(\beta)\right)} - 1$$

Surface velocity of roller

The ratio *Vs/Vo* can be calculated based on the desired strain at $\beta = 0$ or on a target for the average value of $\varepsilon_{\tilde{\alpha}\tilde{\alpha}bndry}$

The normal entry condition

Angle ψ between $\tilde{\alpha}$ and α .

$$\tan(\psi) = \frac{\frac{1}{2}e_{\alpha\beta} + \omega_R}{1 + e_{\alpha\alpha}}$$

Solving for $e_{\alpha\beta}$ and calling it $e_{\alpha\beta bndry}$.

$$e_{\alpha\beta bndry} = 2\tan(\psi)(1 + e_{\alpha\alpha}) - 2\omega_R$$

The value for $e_{\alpha\beta bndry}$ can now be used in the expression for shear strain at the downstream boundary.

$$\varepsilon_{\tilde{\alpha}\tilde{\beta} bndry} = e_{\alpha\beta bndry} + e_{\alpha\alpha}\left(\frac{1}{2}e_{\alpha\beta bndry} - \omega_R\right) + e_{\beta\beta}\left(\frac{1}{2}e_{\alpha\beta bndry} + \omega_R\right)$$

$$+\left(\frac{1}{2}e_{\alpha R} - \omega_\beta\right)\left(\frac{1}{2}e_{\beta R} + \omega_\alpha\right)$$

This is wrong in the paper.

Making the model behave

- Introducing the variable *w* can open Pandora's box, especially when the solution involves compressive stress. Then, wrinkling becomes possible and the PDE becomes unstable.
- So, some way must be found to help the solver.
- One of the things we know in advance is that the MD tension will pull the web toward flatness.
- A real baggy web may not become totally flat. But, for web handling purposes, much can still be learned from a membrane model that forces flatness because a flat web is the ideal.
- So, we force it to be flat and if there are compressive stresses in some parts of the flattened model, we at least know that wrinkles would likely have formed there.

Forcing the web to be flat

$$\left(w - w_{flat}\right)10^6$$

This term is added to the right side of the *z*-direction equilibrium equation.

An expression that defines values of *w* on a plane that touches the four corners of the toroidal segment.

A large number that is determined by trial and error

Numerical analysts call this a penalty function. It forces the difference between w and w_{flat} to become very small to simultaneously satisfy both sides of the PDE.

A better way to think about it is that the 10^6 factor is a large pressure that is forcing the web against a rigid frictionless surface.

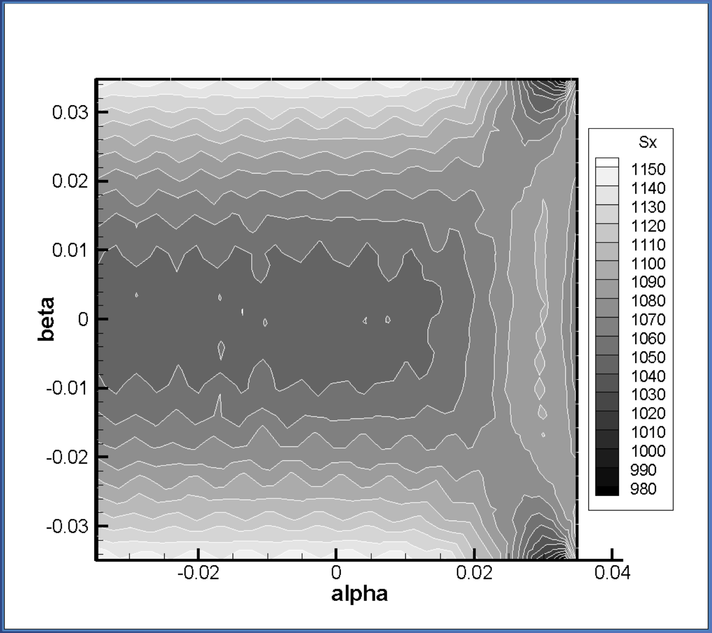
Results - Model parameters

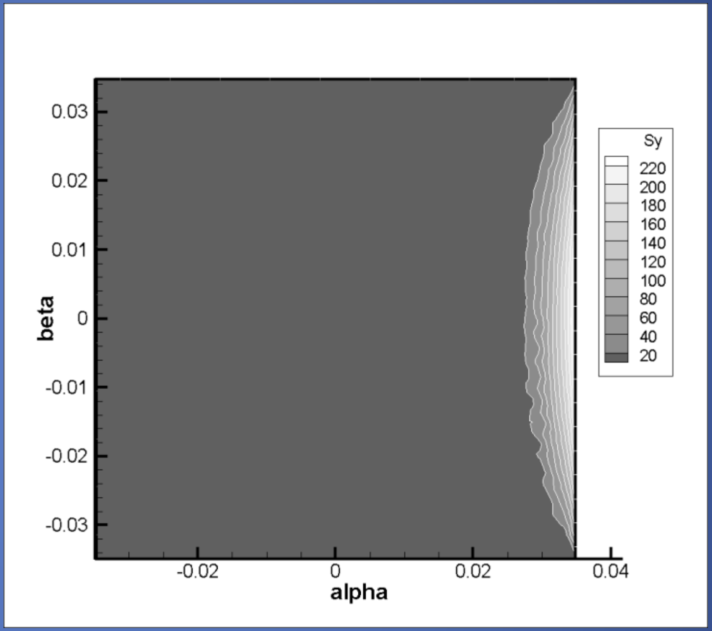
Span length (chord) = 40 inches (1.016 m)
Width (chord) = 20 inches (0.508 m)
Thickness = 0.001 inch (0.025 mm)
Poisson ratio = 0.3
E = 500,000 psi (3.447 Gigapascals)
MD tension = 1000 psi (6.895 Megapascals)
Poloidal angle, β = ± 2 deg (0.035 radian)
Toroidal angle, α = ± 2 deg (0.035 radian)
R_α = 572.7 inches (14.55 m)
R_β = 286.54 inches (7.28 m)

Elliptic curvature:
Difference in MD arc length at center compared to edge = + 0.02%

Hyperbolic curvature:
Difference in MD arc length at center compared to edge = - 0.06%

Results for an elliptic baggy web
Edges shorter than center



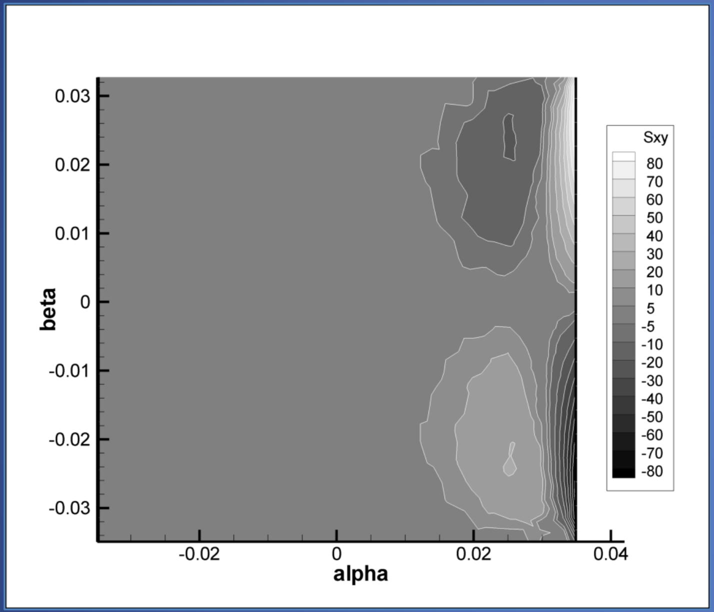


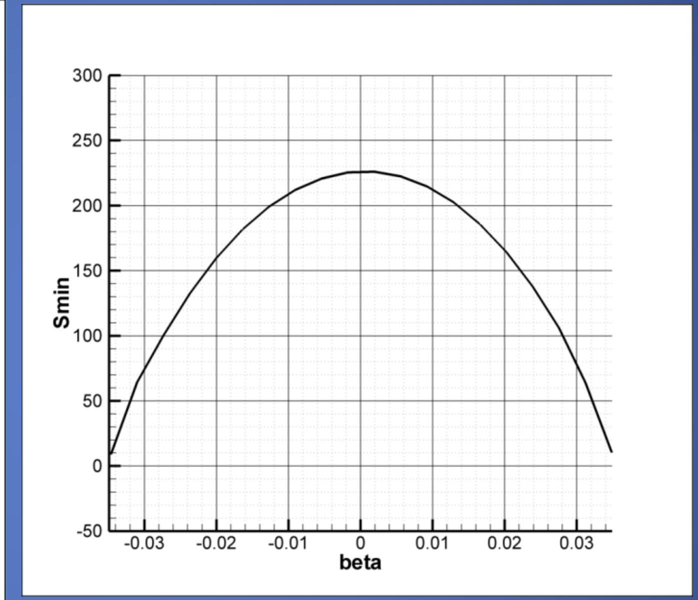
MD stress
Highest stress at edges

CD stress
Positive CD stress at downstream roller

Results for an elliptic baggy web

Edges shorter than center



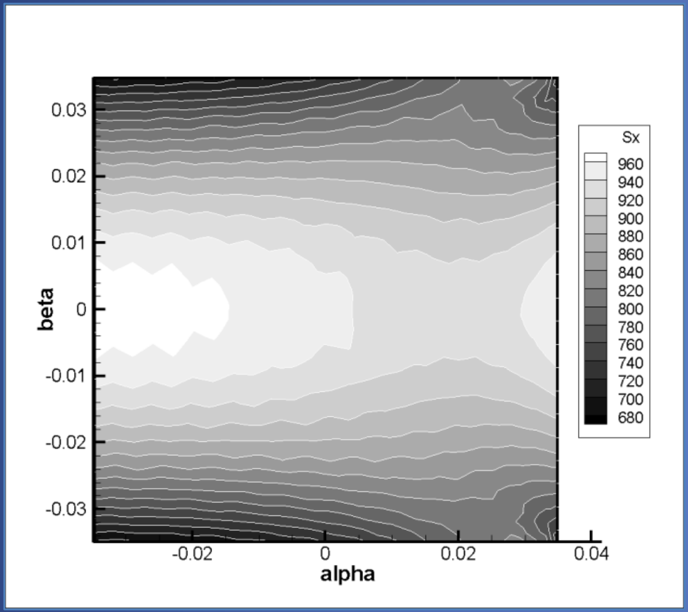


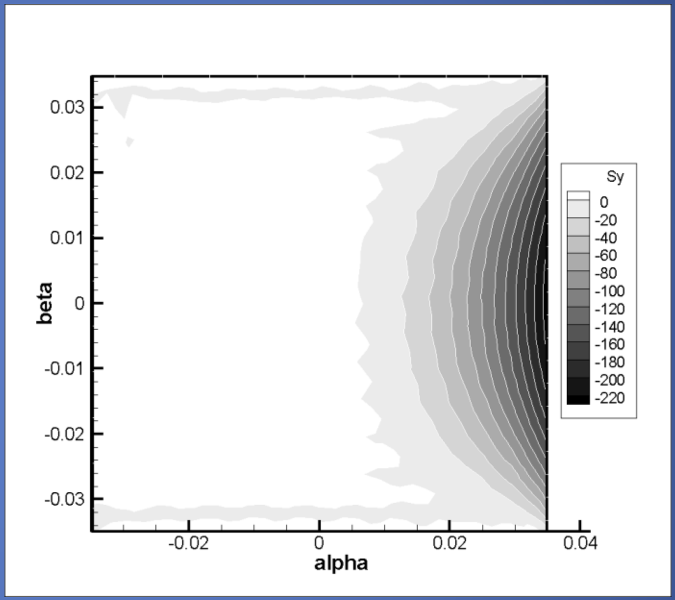
Shear stress

Principal minimum stress at downstream roller
Positive - it spreads

Results for a hyperbolic baggy web

Edges longer than center



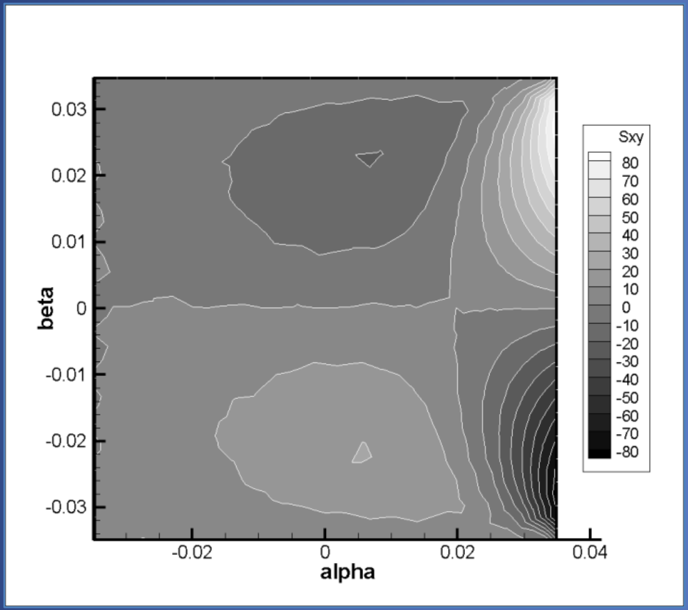


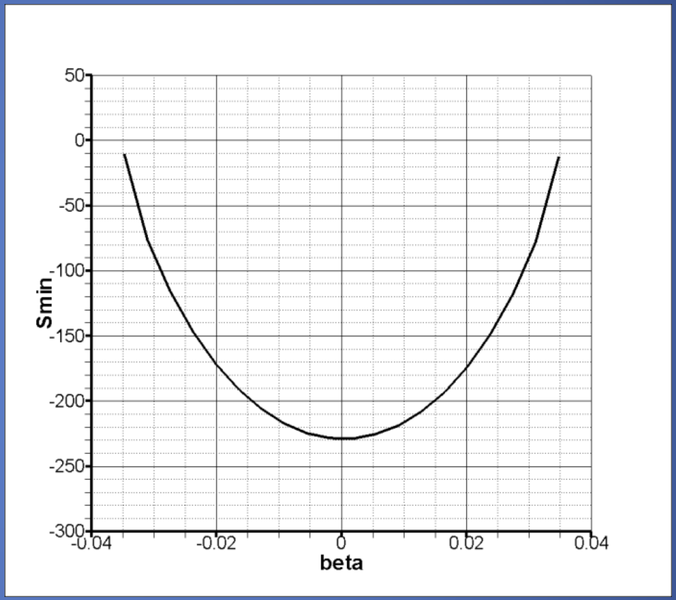
MD stress
Highest stress in center

Negative CD stress
At downstream roller

Results for a hyperbolic baggy web

Edges longer than center



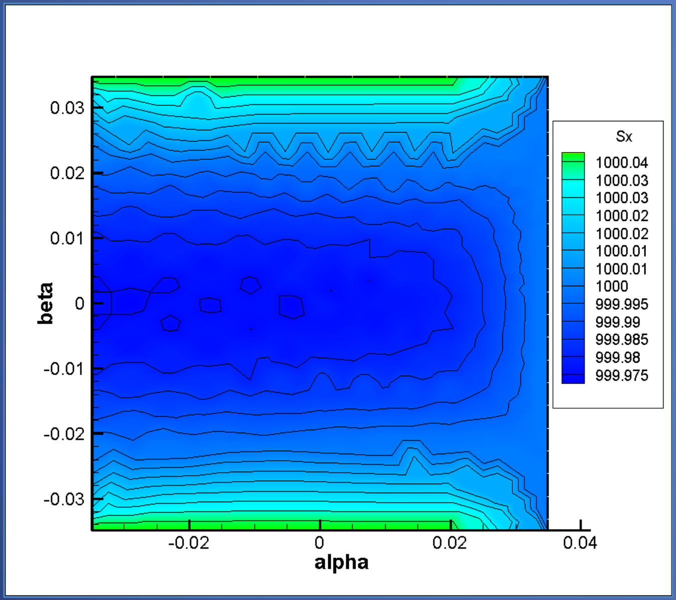


Shear stress

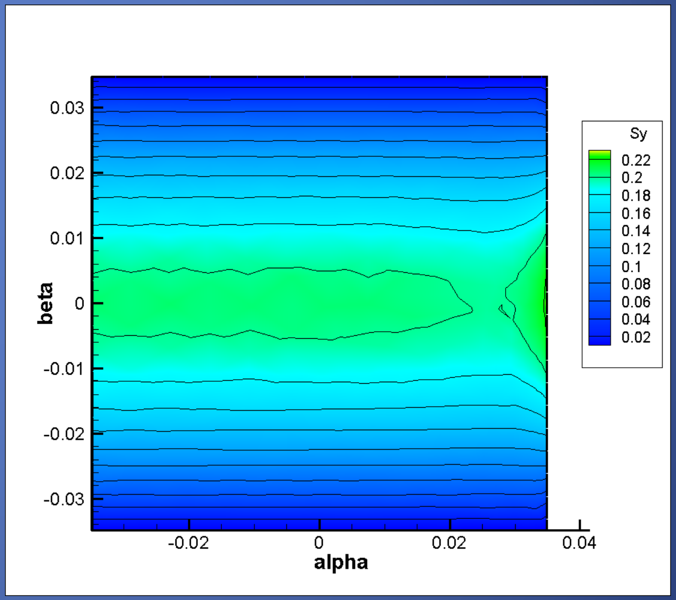
Principal minimum stress at downstream roller
Negative – it wrinkles

Consistency tests

Setting w = 0 is equivalent to having the web run over a frictionless mandrel shaped like the relaxed web. Under that condition, the web should show very little variation in the α-direction stress. A very small amount of β-direction stress would be expected to account for the effect of lateral curvature. Here are the results for the elliptic web with a simple 1000 psi MD load at the downstream roller and no CD load (normal entry turned off).



MD stress



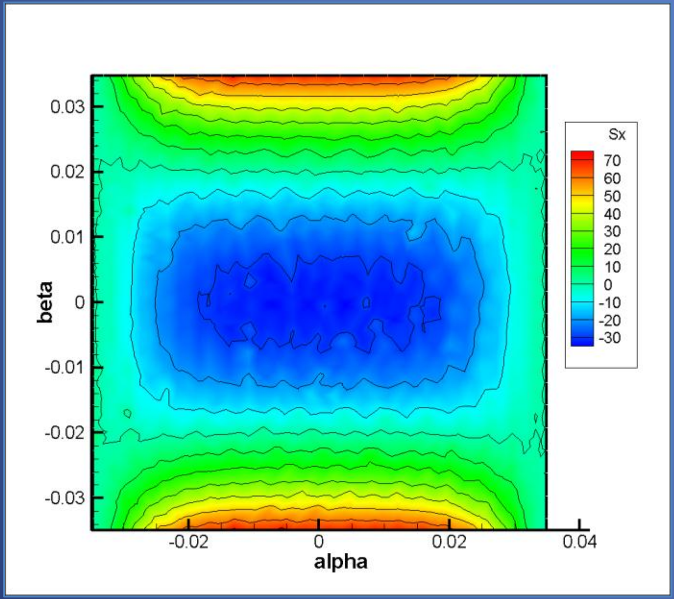
CD stress

Consistency tests

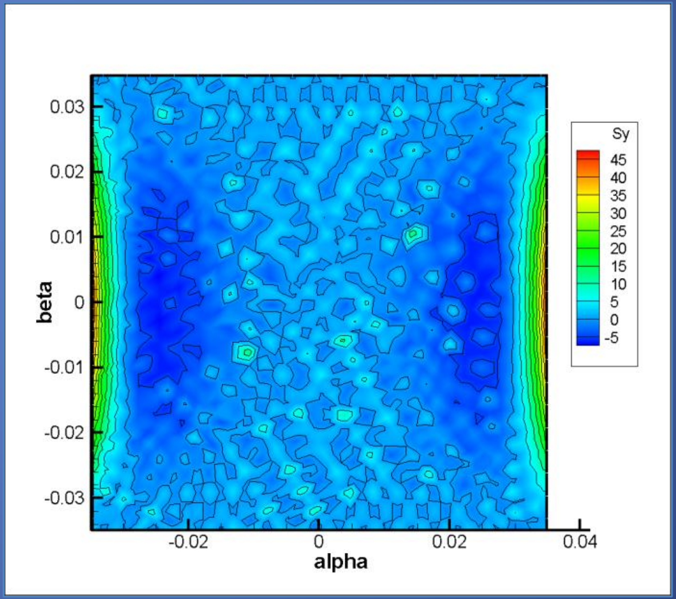
- Another test was to move the w_{flat} plane to a different location. If the model is working correctly, the results shouldn't change. In the elliptic model it was moved from the concave side, where it was touching the corners, to a position where it was tangent to the convex surface. There was no change in the results out to the 5th significant figure.

Consistency tests

After submitting the paper, it occurred to me to ask what the stress fields would look like if all of the edges were left free of stress (free-free) so that only the stresses due to flattening were visible. Here are the results. I think they're persuasive.



MD stress



CD stress

Tentative conclusions
(There is no experimental data)

- An elliptic (short on the edges) baggy web will not wrinkle at a downstream roller. It will develop lateral tensile stress like a uniform web on a concave roller. If the bagginess is clearly visible, it is possible that the lateral tensile stress will be large enough to cause slipping and scratching.
- A hyperbolic (long on the edges) baggy web may wrinkle at a downstream roller. It will develop lateral compressive stress like a uniform web on a crowned roller. If the bagginess is large enough to be observable, it is likely that the compressive stress will be so high that it will be difficult to prevent wrinkling.
- The behavior of elliptic and hyperbolic webs will not change with the direction of wrap.
- Increasing tension to pull out the slack may eliminate gross problems, but it won't change the tendency to spread or wrinkle.
- A web that has narrow baggy lanes due to deep corrugations in a wound roll will likely have spreading where the peaks were and wrinkling at the valleys.

Corrections to the paper

- In the section titled Velocity-Strain equation for a torus, in the first sentence of the first paragraph.
  - If a target value for the average strain is specified, the following procedure ~~can be used to find …~~.
- In equation (63)

$$\varepsilon_{\tilde{\alpha}\tilde{\beta}bndry} = e_{\alpha\beta} + e_{\alpha\alpha}\left(\frac{1}{2}e_{\alpha\beta bndry} - \omega_R\right) + e_{\beta\beta}\left(\frac{1}{2}e_{\alpha\beta bndry} + \omega_R\right) + \left(\frac{1}{2}e_{\alpha R} - \omega_\beta\right)\left(\frac{1}{2}e_{\beta R} + \omega_\alpha\right)$$

This should be. $e_{\alpha\beta bndry}$

Q&A

If you are interested in using nonlinear elasticity theory and would like to have working example of an FEA script for a misaligned roller, compatible with FlexPDE. Just send me a note at essexsys.com.

Corrections to paper

In the section titled Velocity-Strain equation for a torus, the first sentence of the first paragraph, should read,

If a target value for the average strain is specified, the following procedure can be used. ~~*to find …*~~.

In equation (63)

$$\varepsilon_{\tilde{\alpha}\tilde{\beta}bndry} = e_{\alpha\beta} + e_{\alpha\alpha}\left(\frac{1}{2}e_{\alpha\beta bndry} - \omega_R\right) + e_{\beta\beta}\left(\frac{1}{2}e_{\alpha\beta bndry} + \omega_R\right) + \left(\frac{1}{2}e_{\alpha R} - \omega_\beta\right)\left(\frac{1}{2}e_{\beta R} + \omega_\alpha\right)$$

This should be. $e_{\alpha\beta bndry}$