THE CONNECTION BETWEEN LONGITUDINAL AND LATERAL WEB DYNAMICS

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THE ELUSIVE ENTRY ANGLE

Where does the entry angle come from? This paper will show that the answer to this question reveals a connection between longitudinal and lateral behavior that has gone largely unnoticed.

In beam models, entry angle refers to the angle between the tangent to the web centerline and the normal to the roller axis at the line of entry onto the roller. Whenever the entry angle becomes non-zero, a web that is moving longitudinally through a process will also move laterally on the roller in a direction that returns the entry angle to zero. If the web is modeled as a perfectly flexible string, this behavior is intuitively obvious because it bends sharply on entering a roller that is pivoted or shifted laterally. However, in the case of the most commonly used Euler-Bernoulli (E-B) beam model, the web can't make a sharp bend. If it is initially perpendicular to the roller axis, beam theory says that, provided there is no slipping, it should remain perpendicular as the roller is shifted or pivoted and thus wouldn't move. We know from experience, however, that a real moving web begins to move laterally soon after a roller pivots or shifts? So, how can this be?

NOMENCLATURE

A	cross sectional area of web
E	elastic modulus
G	shear modulus
h	thickness of web
I	area moment of inertia
L	span length
m	mass per unit length
n	Shear factor for Timoshenko beam
s	Laplace variable
t	time
T	tension in units of force
V_o	web velocity in machine direction
x	distance along length of web

y lateral displacement of web
y_0 lateral web displacement at upstream roller, relative to ground
y_L lateral web displacement at downstream roller, relative to ground
z lateral displacement of roller relative to ground
θ_L angle between web plane and plane of roller motion at entry to roller
θ_0 angle between web plane and plane of roller motion at exit of roller
θ_r angle of roller axis
β boundary defect angle
ρ density
ϕ rotation of cross section (bending angle)
ψ shear angle
$_0$ subscript indicating value of variable at $x = 0$
$_L$ subscript indicating value of variable at $x = L$

The normal entry equation

The entry angle enters into lateral dynamic analysis through equation (1)[1]. It is commonly referred to as the normal entry equation. Other less-used names are the roller climbing equation, steering equation, parallel entry equation and velocity equation. It defines the lateral velocity of the web dy_L/dt at the line of first contact with a roller[2]. The entry angle is the quantity inside the parenthesis on the right side of equation (1). It is the difference between the roller angle, θ_r and the web slope $dy_L/d\,x$. The circumferential surface speed of the roller is V. The lateral position of the web relative to the roller is y_L and z is the roller position relative to ground.

$$\frac{dy_L}{dt} = V\left(\theta_r - \frac{dy_L}{dx}\right) + \frac{dz}{dt} \qquad (1)$$

The subscripts L and 0 on variables denote their value at the line of entry $(x = L)$ and at the exit of the upstream roller $(x = 0)$, respectively.

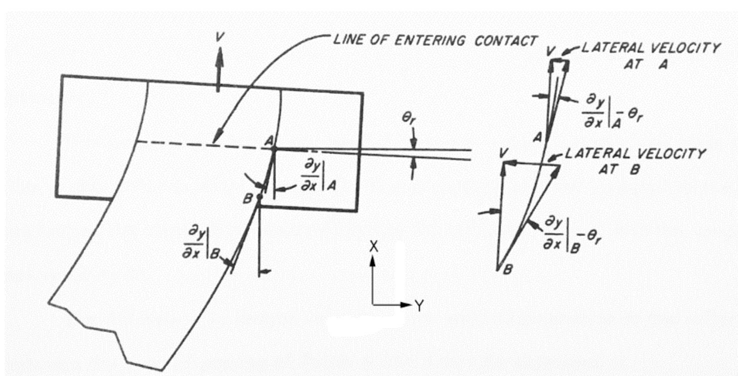

Figure 1
Illustration of normal entry and curvature transport in Shelton's dissertation

[1] Valid only at the line of entering contact provided there is enough traction for the web to adhere to the roller surface without slipping at that location.

[2] To be strictly accurate, dy_L/dt is the lateral velocity of the intersection of *streamlines* of web particles and the line of entering contact.

Slope

Slope is calculated from a <u>static</u> analysis of web shape [1][3]. Equation (2) shows the equation for lateral position *y(x)* that results when face angles[4] ϕ_L, ϕ_0 and lateral displacements, y_L, y_0 are chosen as boundary conditions. The shape factors, g_4, g_5 *and* g_6 depend on span dimensions, mechanical parameters of the web and distance along the span.

It is tempting to think that face angles have been chosen as boundary conditions because they match the roller angles. The main point of this paper hinges on the fact that this is not the case, but for the moment, we have no reason to assume otherwise and it is instructive to see where this assumption leads.

$$y(x) = y_0 + \left(y_0 - y_L\right) g_4\left(x,K,L\right) + \phi_L g_5\left(x,K,L\right) + \phi_0 g_6\left(x,K,L\right) \tag{2}$$

From this, expressions for slope and curvature are developed. The equation for slope at the entry to a downstream roller ($x = L$) is,

$$\frac{dy_L}{dx} = \left(y_0 - y_L\right)\frac{h_1(K,L)}{L} + \phi_L h_2(K,L) + \phi_0 h_3(K,L) \tag{3}$$

The factors, h_1, h_2 *and* h_3 depend on span dimensions and mechanical parameters of the web and whether shear is included.

The face angle ϕ_L in (3) is set equal to the roller angle θ_r. Making this change and substituting (3) into the normal entry equation, the following relationship is obtained.

$$\frac{dy_L}{dt} = V_o\left[\theta_r - \left(y_0 - y_L\right)\frac{h_1(K,L)}{L} - \theta_r h_2(K,L) - \phi_0 h_3(K,L)\right] + \frac{dz_L}{dt} \tag{4}$$

For an Euler-Bernoulli (E-B) beam, h_1 and h_3 are zero and h_2 is unity at $x = L$. So, if a web is initially in a state of uniaxial tension and the downstream end is suddenly pivoted (without lateral shifting) through an angle θ_r, equation (4) becomes,

$$\frac{dy_L}{dt} = 0 \tag{5}$$

Thus, if the web is initially perpendicular to the roller axis, it will never move laterally on the roller. If there is nothing wrong with the shape analysis (and that's been around for a while now), there is no obvious reason to think that this naïve solution wouldn't be correct.

Even though something is clearly missing, it is interesting to see what the full length of the span looks like for typical E-B and Timoshenko versions of this model. This is done by solving equation (4) for y_L (it won't be zero when the effect of shear is included) and substituting into (2). To keep things simple, it will be assumed that the upstream span is in a state of uniform uniaxial stress. So, y_0 and ϕ_0 are set to zero.

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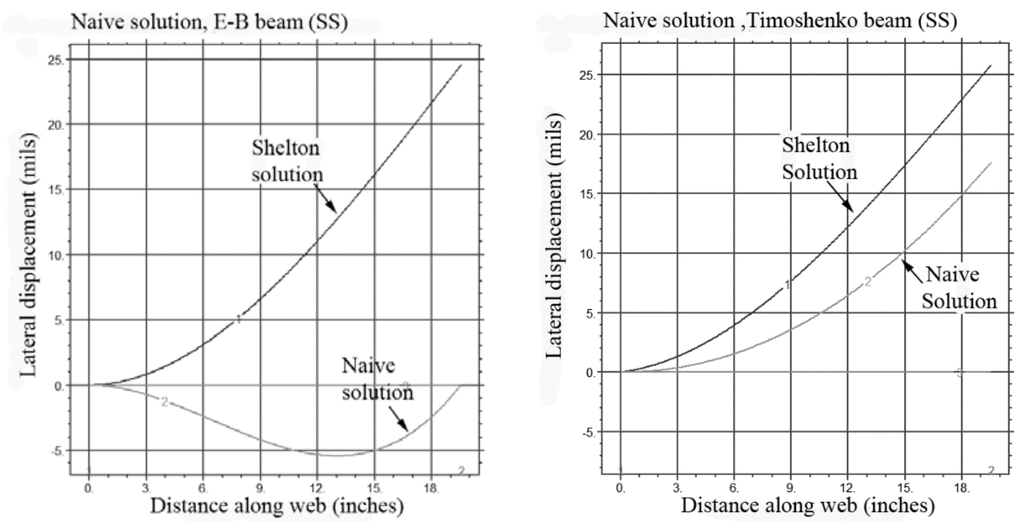
[3] See appendix A for details

[4] The face is defined as a plane that is perpendicular to the web centerline when the web is in a relaxed state. The face angle is the angle between a normal to this plane and the *x*-axis after forces are applied. It is also called the bending angle. For models without shear, the face angle is equal to the web slope. The shear angle, ψ is the angular contribution to slope from shear deformation.



(a) E-B beam (b) Timoshenko beam

Figure 2
Naïve models

The curves in Figure 2 show the shape of the web after it has reached steady state and compares it with the Shelton model (which agrees with experiment). In Figure 2(a) the downstream end of the E-B beam pivoted with the roller but didn't move laterally. In 2(b), which includes shear deformation, the downstream end has moved laterally, but it falls significantly short of the Shelton model.

So, what is missing?

SHELTON'S METHOD: CURVATURE TRANSPORT

Shelton thought about the problem just described. In his seminal dissertation on lateral web dynamics [2] he argued that the entry angle changes because slope variation due to upstream web curvature is transported onto the roller by the web's longitudinal motion (pages 102-104). This is illustrated in Figure 1 (a reproduction from the dissertation)[5]. He says, "The difference in the lateral velocity of the web edge relative to the roller *between the time of passing of Points A and B* may be expressed as"

$$\left.\frac{dy_L}{dt}\right|_A - \left.\frac{dy_L}{dt}\right|_B = V\left(\left.\frac{\partial y}{\partial x}\right|_A - \left.\frac{\partial y}{\partial x}\right|_B\right) \tag{6}$$

Then, he divides both sides by the time, Δt, that it takes for A to move to B. On the right side Δt is replaced by the equivalent quantity $\Delta x/V$. As points A and B are moved infinitesimally close together the left side becomes the lateral acceleration relative to the roller and the right side

[5] When reading some of the references in the bibliography it is important to keep in mind that Shelton followed Timoshenko's sign convention and used a left-handed coordinate system based on the assumption that the positive z-axis is directed out of the page toward the reader. Brown, Benson and Sievers used right-handed coordinates.

becomes the product of V^2 and curvature. Adding d^2z/dt^2 to account for roller motion produces equation (7), known as the acceleration equation

$$\frac{d^2 y_L}{dt^2} = V^2 \frac{d^2 y_L}{dx^2} + \frac{d^2 z}{dt^2} \tag{7}$$

This is used as one of two dynamic boundary conditions (the normal entry equation is the other) in a second order solution based on the shape equation for curvature.

The problem with curvature transport is that it produces an expression for lateral acceleration that conflicts with the definition obtained by simply differentiating the normal entry equation. Shelton was aware of this and made the following comment about it.

> *"Note that Equation 4.1.5 [the acceleration equation] is not merely the derivative of Equation 4.1.2 [the normal entry equation]; differentiation of the latter equation results in an extra term containing the velocity of roller swivelling, $d\theta/dt$. Because of the assumption that shear deflection is negligible, no acceleration can occur as an instantaneous result of roller swiveling. But only indirectly as the web curvature changes. A suddenly swivelling roller instantaneously swivels the downstream end of the web an equal amount, so that no instantaneous change in steering rate occurs, in contrast to the first-order theory of Chapter III [which employs a flexible string model of the web]."*

Still, there can be only one value for lateral acceleration at any instant and if the normal entry equation is valid, there is no reason to think that its time derivative wouldn't provide it.

Regardless of any concern about it, Shelton showed in his dissertation [2] that using equation (7) in a dynamic model produced excellent agreement with experiments. He tested four configurations; a parallel pair with $KL = 2$, a parallel pair with $KL = 10$, an oversteering guide and an understeering guide. Amplitude and phase response were measured in each case at six different frequencies. All had long spans in which the effects of shear were insignificant.

BENSON'S METHOD: THE MATERIAL DERIVATIVE

Benson, in a 2002 paper [3], found a better way to derive the acceleration equation. He started by assuming that the pivoting *velocities* of the roller angle, θ_r, and web face angle, ϕ_L, must match at the line of entry of the web onto the roller. He then applied the material derivative and arrived at the following expression.

$$\frac{d\theta_r}{dt} = \frac{D\phi_L}{Dt} = \frac{d\phi_L}{dt} + V\frac{d\phi_L}{dx} \tag{8}$$

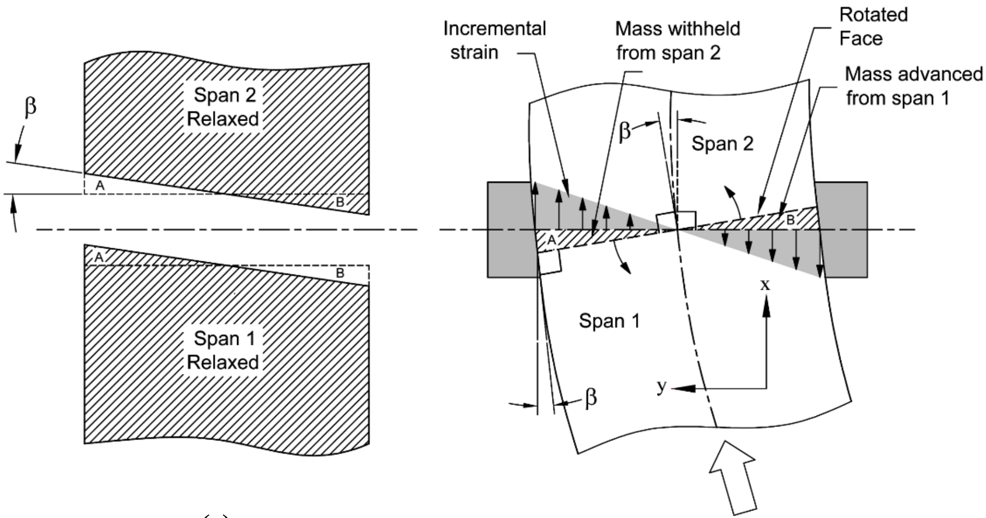
Benson chose to organize his model in the form of four first-order equations. So, he wasn't interested in anything like acceleration equation (7) as a boundary condition. Nevertheless, to help establish the validity of his model, he showed that equation (8) could be used to derive it. The result of his derivation, shown in equation (9) below, included the effect of shear and is in agreement with Brown [4].

$$\frac{d^2 y_L}{dt^2} = V^2 \frac{d^2 y_L}{dx^2} + \frac{d^2 z_L}{dt^2} - V\left(\frac{d\psi_L}{dt} + V\frac{d\psi_L}{dx}\right) \tag{9}$$

When shear ψ_L is eliminated, equation (9) defaults to Shelton's acceleration equation (7).

THE ENTRY ANGLE IS CAUSED BY LONGITUDINAL TRANSFER OF MASS

In a 2017 IWEB paper [4], I showed that the entry angle of the normal entry equation (1) is entirely due to the effect of mass transferred longitudinally between spans. I won't repeat the details of the analysis of that paper, but will, instead, summarize the principle results.



(a) (b)

Figure 3
Effect of mass transfer after roller shift
Without shear (Euler-Bernoulli model)

In Figure 3(b), the web is shown shortly after a moment of force comes into existence at the entry to the roller. In the case shown here, a sudden lateral shift of the roller to the left produced curvature with its center at some point far to the right. The curvature created a tapered, incremental strain profile which is positive on the left side and negative on the right. The positive incremental strain on the left caused a decrease in mass flow from span 1 to 2 relative to the flow at the center. This, in turn, caused an accumulation of mass in span 1, shown as the wedge-shaped area A. Area A also represents the deficit of mass in span 2. The negative incremental strain on the right caused an increase in mass flow from span 1 to 2 and created the wedge-shaped accumulation of mass in span 2 labeled B. Area B also represents the deficit of mass in span 1.

The net effect is angular rotation of the face of the web at the line of entry through an angle, β. There is no slipping involved in the formation of β. It is entirely due to variations in mass flow that change the relationship between face angle and roller angle from $\phi_L = \theta_r$ to $\phi_L = \theta_r + \beta$. Analysis of the relationship between the strain profile and mass flow shows that β is defined as,

$$-\frac{d\beta}{dt} = \frac{d}{dt}\left(\theta_r - \frac{dy_L}{dx}\right) = V\frac{d^2 y_L}{dx^2} \tag{10}$$

When the angle β begins growing, the web begins moving relative to the roller in a direction that reduces the moment that caused it. It will continue to move until the entry angle becomes zero.

For a Timoshenko model with shear, relationship (10) becomes,

$$-\frac{d}{dt}\left(\beta + \psi_L\right) = \frac{d}{dt}\left(\theta_r - \frac{dy_L}{dx}\right) = aV\frac{d^2 y_L}{dx^2} - \frac{d\psi}{dt} \tag{11}$$

And

$$a = 1 + \frac{nT}{AG} \tag{12}$$

where n is the shear factor, T is the longitudinal tension, A is the cross-sectional area of the web and G is the shear modulus.

When shear is included in the model, the face between the two spans still rotates as described above, but β increases by the factor, a, and the entry angle becomes $-\beta - \psi$.

It is fair to ask why mass transfer is needed. After all, Shelton's and Benson's equations aren't wrong. The answer is that it is an essential part of the physical picture that has been missing and, as will be shown in a companion paper, it is the key to understanding how to combine lateral and longitudinal behavior in a single model.

Connections with the methods of Shelton and Benson

Since Shelton's E-B model was confirmed by experiment and Benson's material derivative can be used to derive Shelton's acceleration equation, it should not be surprising to find that they are both mathematically equivalent to the mass transfer model.

Equating the two values of acceleration from Shelton's model (acceleration of equation (7) and the time derivative of velocity from the normal entry equation (1)) produces the second equality of expression (10).

Substituting $dy_L/dx = \phi_L$ (true for an E-B beam) in Benson's material derivative (8) also produces the second equality of expression (10).

Why did Benson's velocity matching work?

In its relaxed state, all the particles in a uniform web are assumed to be moving in straight lines aligned with the x-axis. As the web deforms, those paths become curved to conform with the web shape. It is important to realize, however, that in a moving web, the particles following those paths will not all be travelling at the same speed. For example, particles on the outside edge of a curve, and not in proximity to a roller, will be travelling faster than particles on the inside edge. Then, when they arrive at the roller, where, it is assumed, they will "stick" to its surface, they must take on its velocity. That velocity must be the same at all points along the line of entry. Benson recognized this fact in his velocity matching boundary condition when he said, "It is further expected that the web will stick to the roller for all points of first contact – not just at the web's centerline. To achieve that, we must also match the rotational velocities of the roller and the web." That can only happen if the rate of mass flow changes. So, although he made no mention of it, velocity matching at the roller effectively engages the mathematics of mass transfer.[6]

Limitations of beam theory models

Beam theory, because of its 1-dimensional nature, accommodates only simple strain profiles produced by moments due to pivoting or shifting of rollers. There are other interesting problems like concave roller applications that can only be solved with 2-D numerical methods like those described in "Effects of Concave Rollers, Curved-Axis Rollers and Web Camber on the Deformation and Translation of a Moving Web" [5].

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[6] I owe Dilwyn Jones a debt of gratitude for reviewing early versions of this paper and patiently defending Benson's method. I was inclined to distrust anything that didn't explicitly mention mass conservation, but he convinced me of its validity, using an argument like the one I just made.

CONCLUSION

Mass transfer[7], in the form of the continuity equation, has been part of tension analysis for decades, but it has not been used explicitly in the analysis of lateral behavior. It is now clear, however, that it is a vital part of the conceptual framework for both subjects.

APPENDIX A

WEB SHAPE EQUATIONS FOR A BEAM MODEL

The Elastic Curve

Shelton was the first to use beam theory in models of lateral web dynamics. He derived equations for the elastic curve of single spans using both Euler-Bernoulli and Timoshenko beam theories [6, 7]. The method presented here is due to Lisa Sievers and is particularly suited to multi-span problems [8]. It begins by first observing that the bending and shear angles are additive. The face is defined as a plane that is perpendicular to the web centerline when the web is in a relaxed state. The face angle ϕ is the angle between a normal to this plane and the x-axis after forces are applied. It is also called the bending angle. For models without shear, the face angle is equal to the web slope. The shear angle, ψ is the angular contribution to slope from shear deformation.

$$\frac{dy}{dx} = \phi + \psi \tag{13}$$

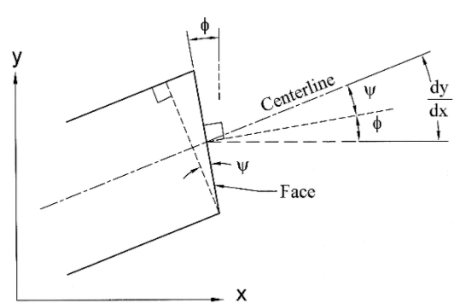

Figure 4

Relationship of Slope, shear angle and bending angle

$\frac{dy}{dx}$= slope , ψ= shear angle , ϕ=bending angle

She then applied Hamilton's principle [9] to derive the equations of motion.

This produces a solution that includes both time and spatial derivatives. The time derivatives are useful in determining the potential effect of natural vibrations. She found that the separation between the natural frequencies of the web and frequencies of interest in typical applications, while not as great as one might expect, are usually adequate to safely ignore the time-related

[7] And its close cousin, transport of strain.

terms. Details may be found in several references [8, 10, 11]. When the time-related terms are removed, the following two equations are left,

$$\left(1+\frac{nT}{AG}\right)\frac{d^2y}{dx^2}-\frac{d\phi}{dx}=0 \tag{14}$$

$$EI\frac{d^2\phi}{dx^2}+\frac{AG}{n}\left(\frac{dy}{dx}-\phi\right)=0 \tag{15}$$

These relationships can be manipulated to obtain the same fourth order differential equation found by Shelton.

$$\frac{d^4y}{dx^4}-K^2\frac{d^2y}{dx^2}=0 \tag{16}$$

where,

$$K^2=\frac{T}{EI\left(1+\frac{nT}{AG}\right)} \tag{17}$$

The solution to (16), familiar to all web handling researchers, is

$$y(x)=C_1\sinh(Kx)+C_2\cosh(Kx)+C_3x+C_4 \tag{18}$$

The solution just described applies to a Timoshenko beam model that includes the effects of shear deformation. It defaults to the Euler-Bernoulli (E-B) beam model if the shear factor *n* is set to zero.

Boundary Conditions

In this model, as in all other multi-span models to-date, the interaction of the web with rollers is greatly simplified. The width of the contact zone in the process direction is assumed to be zero.

Four boundary conditions are required. Lateral position at the upstream and downstream rollers provide two of them.

Sievers believed that the bending angle was a better choice than slope for a boundary condition because it would be continuous across rollers, while the slope and shear would be discontinuous. This is incorrect. The effect of wrap angle on rollers causes the bending angle to be discontinuous. However, the choice of bending angle for a boundary condition is advantageous for another reason. It is directly related to the roller angle (or, more precisely, its projection on the plane of the web).

Expressions for shear angle ψ and bending angle ϕ are derived from equations (13), (14) and (15).

$$\psi=-EIa\frac{n}{AG}\frac{d^3y}{dx^3} \tag{19}$$

where

$$a = 1 + \frac{nT}{AG} \tag{20}$$

and,

$$\phi = \frac{dy}{dx} + EIa\frac{n}{AG}\frac{d^3y}{dx^3} \tag{21}$$

So, the boundary conditions of the Timoshenko beam model will be,

$$y\big|_{x=0} = y_0 \qquad y\big|_{x=L} = y_L$$

$$\left.\frac{dy}{dx}\right|_{x=0} + EIa\frac{n}{AG}\left.\frac{d^3y}{dx^3}\right|_{x=0} = \phi_0 \qquad \left.\frac{dy}{dx}\right|_{x=L} + EIa\frac{n}{AG}\left.\frac{d^3y}{dx^3}\right|_{x=L} = \phi_L \tag{22}$$

Equation (18) and its derivatives are substituted into the four equations of (22) which are then solved simultaneously for C_1, C_2, C_3 and C_4.

<u>The Static Equation of Web Shape.</u>

Inserting values for C_1, C_2, C_3 and C_4 in (18) and collecting terms,

$$y(x) = y_0 + \left(y_0 - y_L\right)g_4\left(x,K,L\right) + \phi_L g_5\left(x,K,L\right) + \phi_0 g_6\left(x,K,L\right) \tag{23}$$

where,

$$g_4(x,K,L) = \frac{\cosh(Kx)+\cosh(KL)-\cosh(KL-Kx)-Kax\sinh(KL)-1}{KLa\sinh(KL)-2\left(\cosh(KL-1\right)}$$

$$g_5(x,K,L) = \frac{KLa\left(\cosh(Kx)-1\right)-Kax\left(\cosh(KL)-1\right)-\sinh(Kx)-\sinh(KL-Kx)+\sinh(KL)}{Ka[KLa\sinh(KL)-2\left(\cosh(KL-1)\right)]} \tag{24}$$

$$g_6(x,K,L) = \frac{\sinh(Kx)-\sinh(KL)+\sinh(KL-Kx)-KLa\left(\cosh(KL-Kx)-1\right)+Ka(L-x)(\cosh(KL)-1)}{Ka[KLa\sinh(KL)-2\left(\cosh(KL-1)\right)]}$$

Equations (24) are called shape functions.

Following the example of Young, Shelton and Kardimilas (YSK) [12], y_o appears twice in expression (23). This reduces the number of shape functions from four to three.

Two other equations that will be needed later are the first and second derivatives of (23) at $x = L$.

$$\left.\frac{dy(x)}{dx}\right|_L = \left(y_0 - y_L\right)\frac{h_1}{L} + \phi_L h_2 + \phi_0 h_3 \tag{25}$$

$$\left.\frac{d^2y(x)}{dx^2}\right|_L = \left(y_0 - y_L\right)\frac{g_1}{L^2} + \phi_L\frac{g_2}{L} + \phi_0\frac{g_3}{L} \tag{26}$$

where,

$$h_1 = \frac{KLa\sinh(KL)(1-a)}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]}$$
$$h_2 = \frac{(a+1)(1-\cosh(KL))+KLa\sinh(KL)}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]} \qquad (27)$$
$$h_3 = \frac{(a-1)(1-\cosh(KL))}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]}$$

$$g_1 = \frac{K^2L^2a(\cosh(KL)-1)}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]}$$
$$g_2 = \frac{KL(KLa\cosh(KL)-\sinh(KL))}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]} \qquad (28)$$
$$g_3 = \frac{KL(\sinh(KL)-KLa)}{a\left[KLa\sinh(KL)-2(\cosh(KL)-1)\right]}$$

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