Behavior of a Thin Flexible Twisted Web

IWEB 2007

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Essex Systems

A Continuation of Earlier Work

- **A nonlinear PDE model, suitable for use with low-cost FEA software.**
- **Straightforward three-dimensional model is problematic for a low-cost FEA PDE solver.**
- **So, a two-dimensional solution was sought.**
- **Inspiration came from large-deflection plate theory .**

Features of the Model

- **Allows analysis of both in-plane and out-of-plane misalignment, including large rotations.**
- **Based on Novoshilov's equations for small strains and <u>large</u> <u>rotations</u>**
- **Incorporates the normal entry and normal strain Rules**

Can't predict wrinkling

- An ideal model for a twisted web would include bending behavior so that wrinkling can be analyzed.
- Theodore von Karman developed a candidate that looks attractive.
- With nonlinear improvements it might be capable of predicting elastic instability.

Von Karman Model

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) \frac{D}{h} = \sigma_x \frac{\partial^2 w}{\partial x^2} + \sigma_y \frac{\partial^2 w}{\partial y^2} + \sigma_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

Unfortunately, high order derivatives on left side of last equation are problems for low-cost FEA.

Unbuckled Web is Easier

- Curvatures due to twisting are small and the bending moments are insignificant.
- Only buckling produces large curvatures in the form of troughs.
- Therefore, the problem can be separated into two parts – unbuckled and buckled.
- This model treats the unbuckled twisted web.

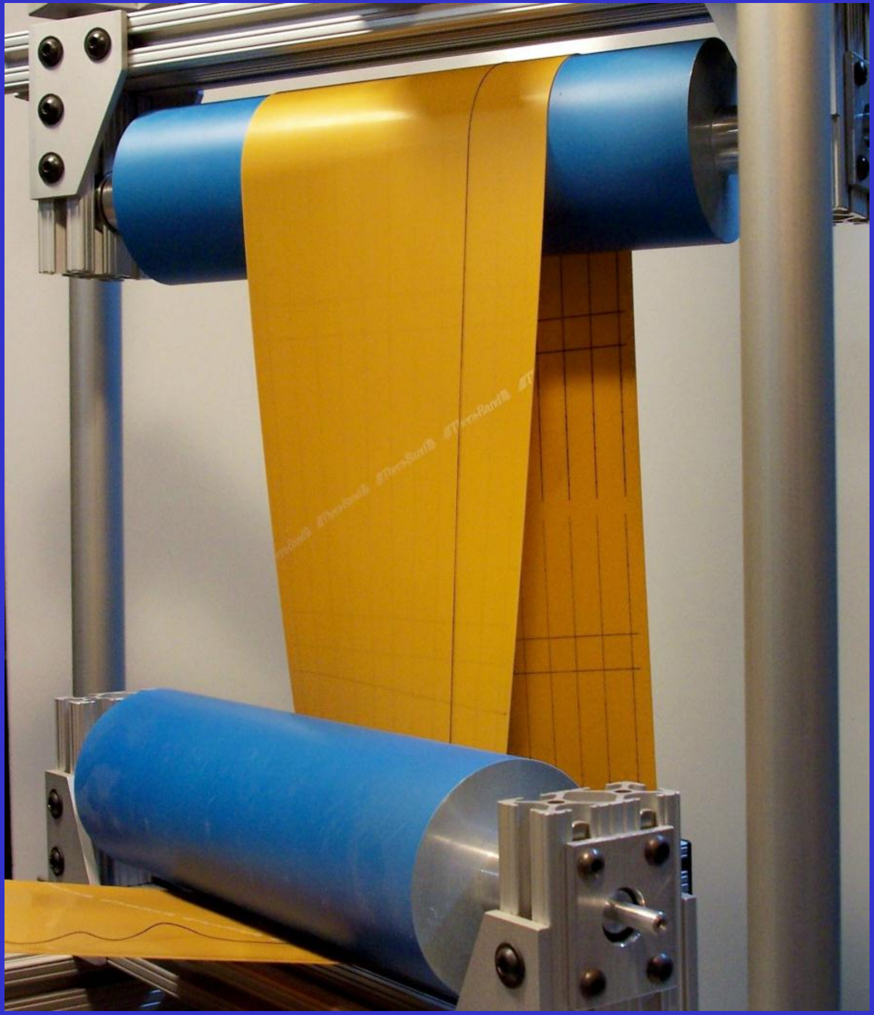
Earlier Work

- Good and Straughan developed a flat model based on an estimate of the MD elongation profile in a twisted web.
  - Used Timoshenko's theory of buckling in a cylindrical shell to predict onset of wrinkling.
  - Did not incorporate normal strain or normal entry conditions.
  - Provided good experimental confirmation

Earlier Work

- Mockensturm developed a fully nonlinear model based on modern plate theory (Naghdi).
  - Difficult mathematics. But, maybe worth the effort.
  - Model predicts onset and shape of troughs.
  - Does not appear to incorporate normal strain and normal entry rules.
    - Conservation of mass mentioned. But, it is unclear how it is enforced.
  - No experimental confirmation.

The Model



- Zero MD curvature.

$$\frac{\partial^2 w}{\partial x^2} = 0$$

Strains

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

- Strains relative to deformed coordinates
- Expressed in undeformed coordinates

Equilibrium Equations

$$\frac{\partial}{\partial x}\left[\left(1+\frac{\partial u}{\partial x}\right)\sigma_{xx}+\frac{\partial u}{\partial y}\sigma_{xy}\right]+\frac{\partial}{\partial y}\left[\left(1+\frac{\partial u}{\partial x}\right)\sigma_{xy}+\frac{\partial u}{\partial y}\sigma_{y}\right]=0$$

$$\frac{\partial}{\partial x}\left[\left(1+\frac{\partial v}{\partial y}\right)\sigma_{xy}+\frac{\partial v}{\partial x}\sigma_{xx}\right]+\frac{\partial}{\partial y}\left[\left(1+\frac{\partial v}{\partial y}\right)\sigma_{y}+\frac{\partial v}{\partial x}\sigma_{xy}\right]=0$$

- σ_{xx} , σ_{yy} , and σ_{xy} are the stresses in the directions of the deformed coordinate system.

Direction Cosines

	i_1 (Deformed x)	i_2 (Deformed y)
X	$\dfrac{1+u_x}{\sqrt{1+2\varepsilon_{xx}}}$	$\dfrac{u_y}{\sqrt{1+2\varepsilon_{yy}}}$
Y	$\dfrac{v_x}{\sqrt{1+2\varepsilon_{xx}}}$	$\dfrac{1+u_x}{\sqrt{1+2\varepsilon_{yy}}}$
Z	$\dfrac{w_x}{\sqrt{1+2\varepsilon_{xx}}}$	$\dfrac{w_y}{\sqrt{1+2\varepsilon_{yy}}}$

Stress Definitions

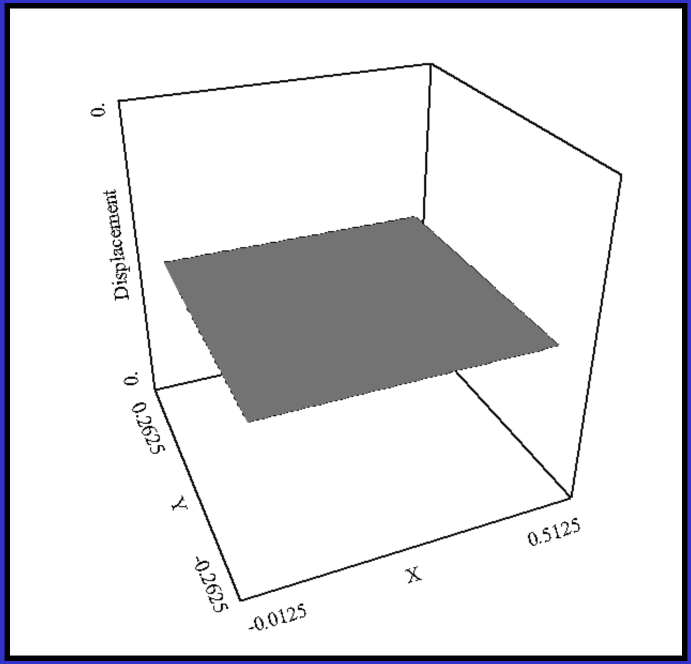
$$\sigma_{xx} = \frac{E}{1-\mu^2}\left(\varepsilon_{xx} + \mu\varepsilon_{yy}\right)$$

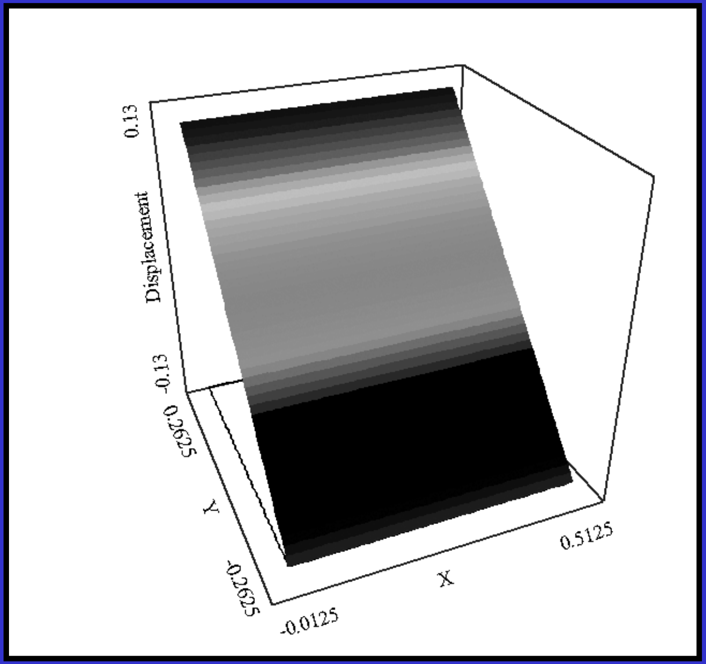
$$\sigma_{yy} = \frac{E}{1-\mu^2}\left(\varepsilon_{yy} + \mu\varepsilon_{xx}\right)$$

$$\tau_{xy} = \frac{E}{2\left(1+\mu\right)}\left(\varepsilon_{xy}\right)$$

- These stresses are referred to deformed coordinate system.
- Subscripts may be confusing.

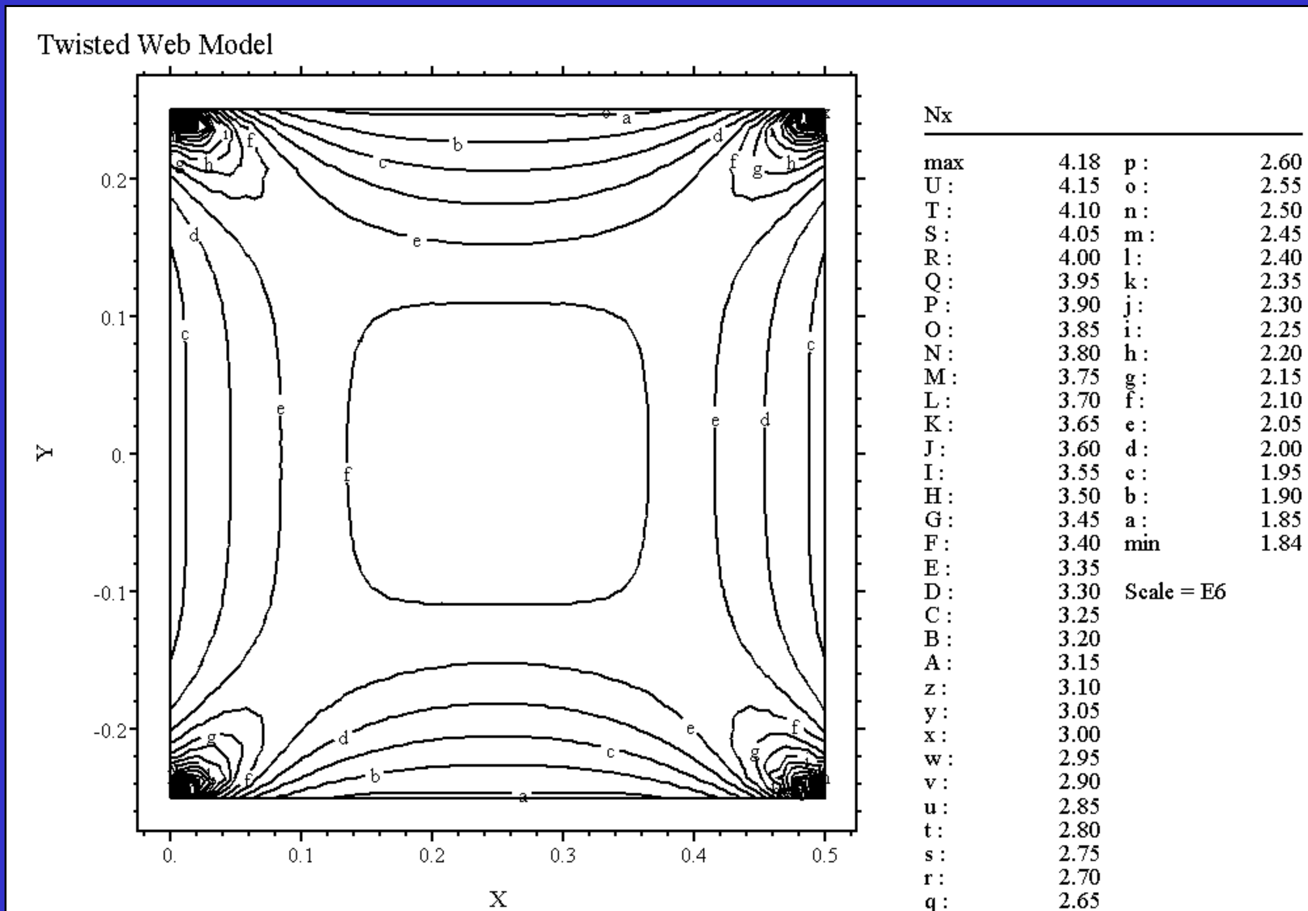
Verification





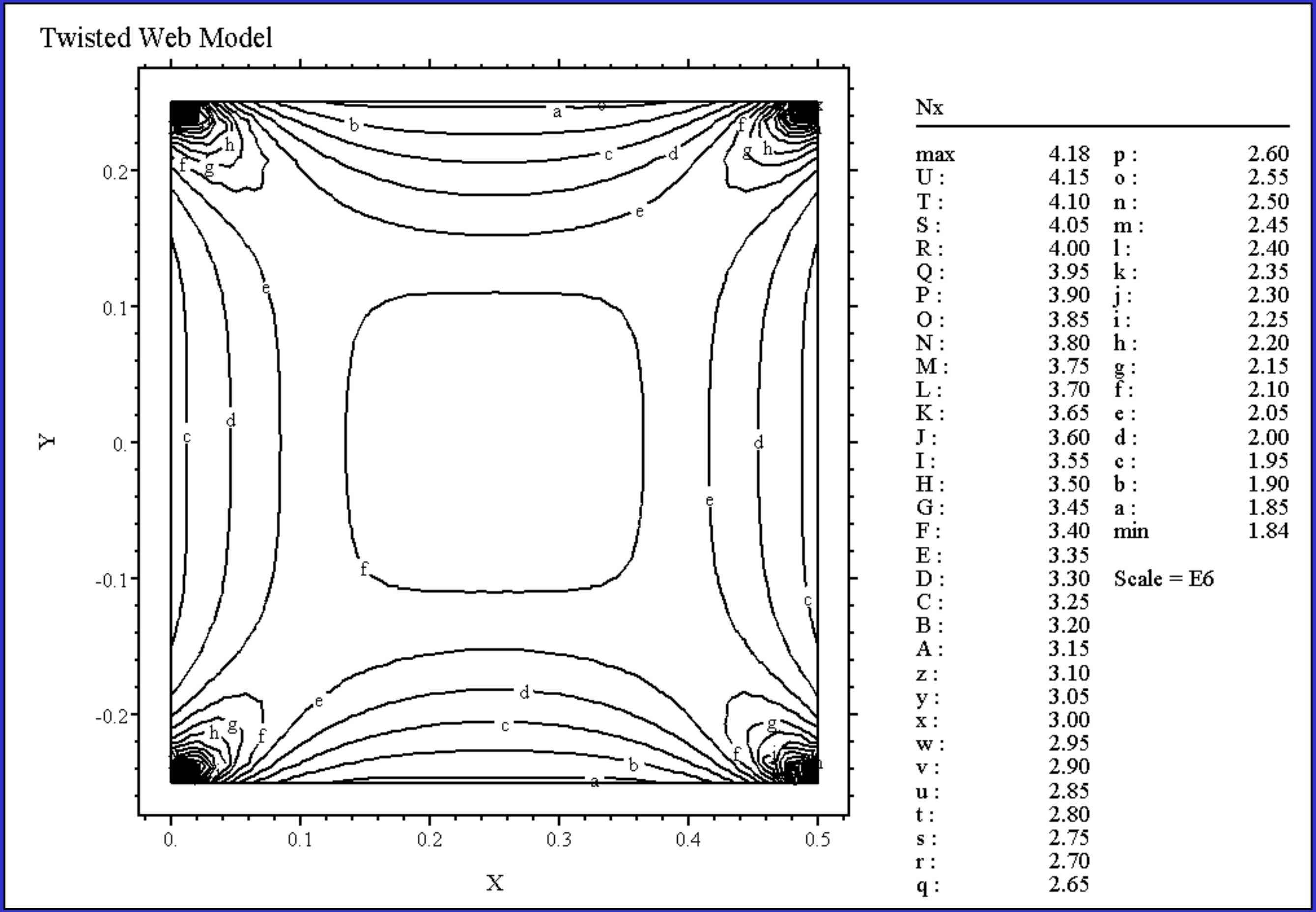
- Stretched a flat membrane and then rotated it as a rigid body. The stresses shouldn’t change.

Horizontal

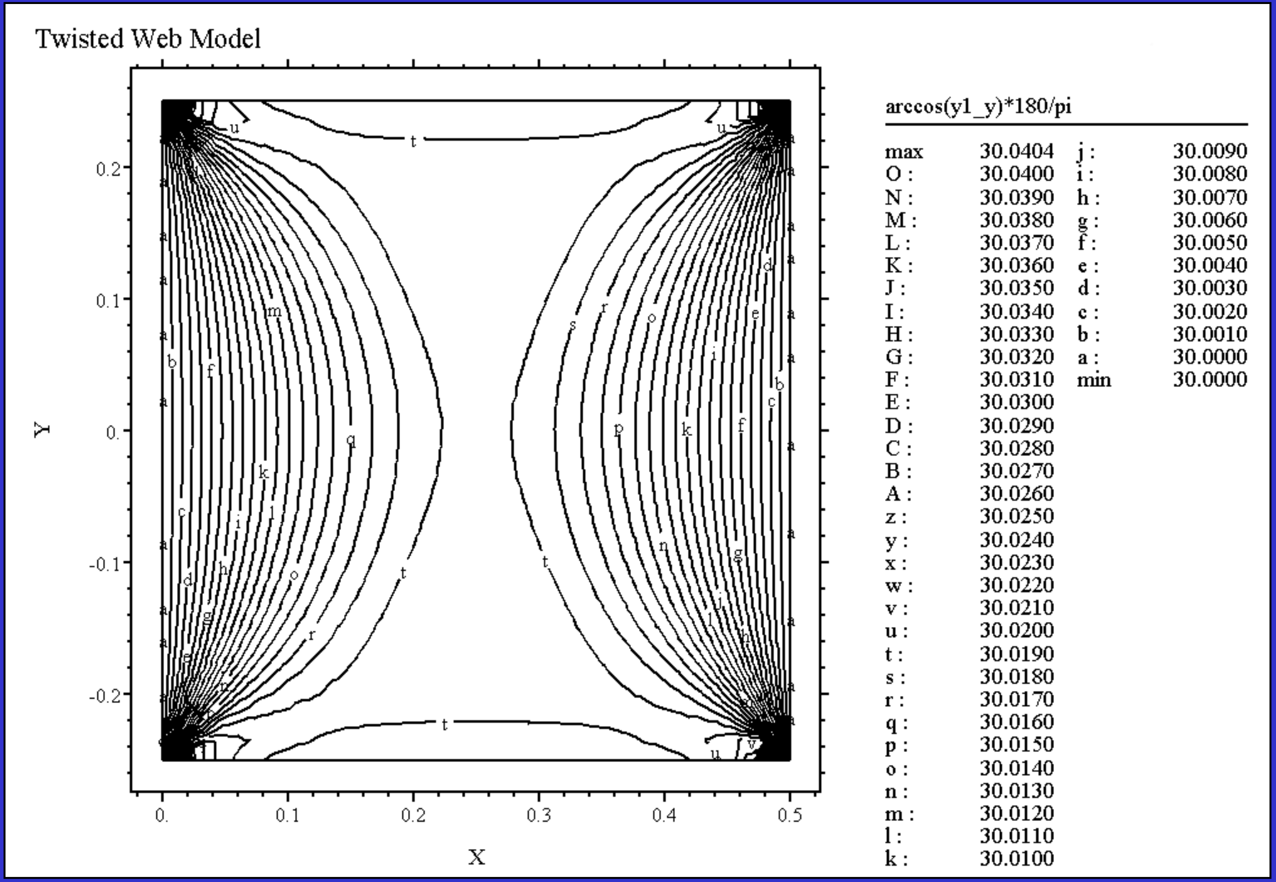


Twisted Web Model
Nx
max 4.18
U : 4.15
T : 4.10
S : 4.05
R : 4.00
Q : 3.95
P : 3.90
O : 3.85
N : 3.80
M : 3.75
L : 3.70
K : 3.65
J : 3.60
I : 3.55
H : 3.50
G : 3.45
F : 3.40
E : 3.35
D : 3.30
C : 3.25
B : 3.20
A : 3.15
z : 3.10
y : 3.05
x : 3.00
w : 2.95
v : 2.90
u : 2.85
t : 2.80
s : 2.75
r : 2.70
q : 2.65
p : 2.60
o : 2.55
n : 2.50
m : 2.45
l : 2.40
k : 2.35
j : 2.30
i : 2.25
h : 2.20
g : 2.15
f : 2.10
e : 2.05
d : 2.00
c : 1.95
b : 1.90
a : 1.85
min 1.84
Scale = E6
Y
0.2
0.1
0.
-0.1
-0.2
0.
0.1
0.2
0.3
0.4
0.5
X

Rotated 30 degrees



Direction cosine



$$\mathrm{Cos}^{-1}\ \frac{1+u_x}{\sqrt{1+2\varepsilon_{yy}}}$$

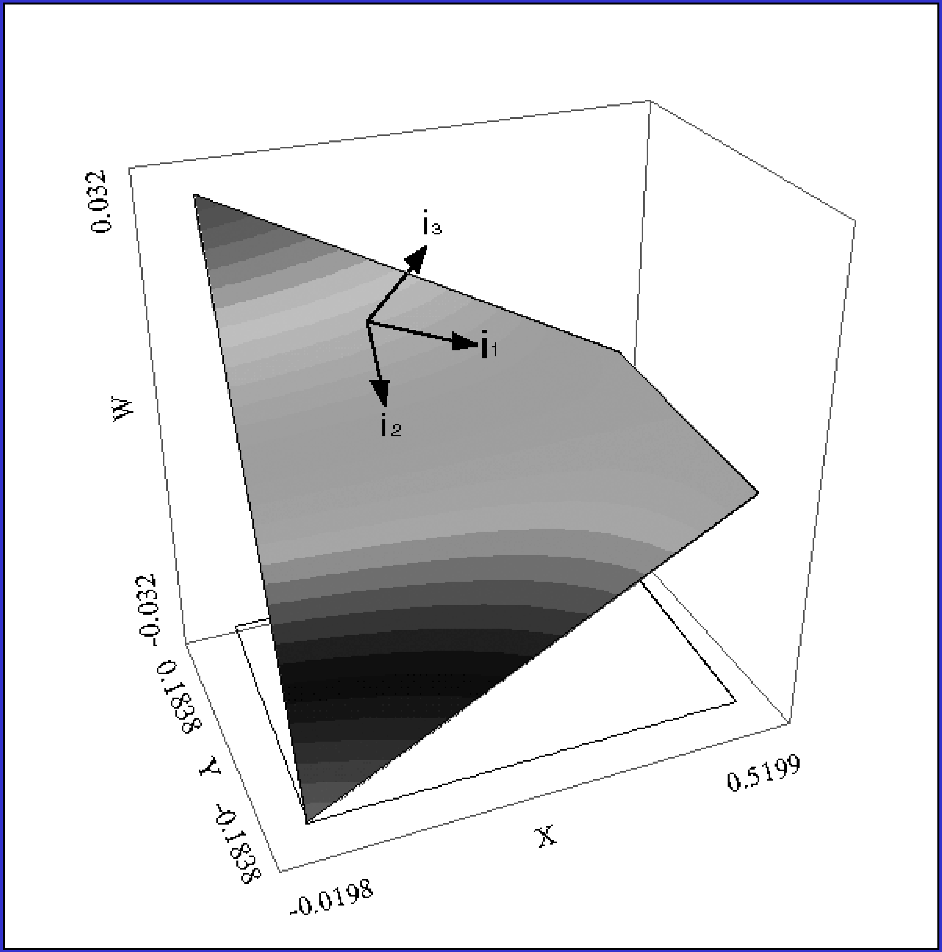
Effect of Twist on Boundary Shape

- Twist causes the roller wrap to increase at one edge and decrease at the other.
- Relaxed shape a parallelogram.
- CD curvature of the boundaries, due to the helical path on the roller surfaces is not taken into account. It’s very small.
- Iterative procedure for finding intersection of web with roller available on request.

Boundary Defect

- It is usually assumed that the relaxed shape of the web is a thin rectangular sheet. However, the boundaries that we see when looking at a process line apply to the web AFTER it has been stressed and distorted. So the shape of the relaxed web, which is actually the thing being analyzed, is not really known in advance.
- Not very important in twisted web. But, FEA solver makes the correction easy. So, it will be included

Boundaries



- Twisting is done at the upstream boundary .
- Previous span assumed to be in a state of pure MD stress and rotated (without twisting) into alignment with the upstream boundary

Boundary Conditions

- Generally need three at each edge.
- Upstream roller
  - x-displacement, $u = 0$
  - y-displacement same as Poisson contraction in previous span plus rotation.
  - z-displacement to rotate web in combination with y-displacement.
- The y displacement is tricky

Upstream condition for *v*

- The Poisson contraction in the previous span is related to *v* by the nonlinear strain relation described earlier.

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = -\mu\varepsilon_o$$

- ε_o is the x-axis strain in the previous span.
- This is treated as an ordinary differential equation and solved for *v*. The y derivatives of *u* and *w* are zero in the upstream span.

Upstream condition for *v* & *w*

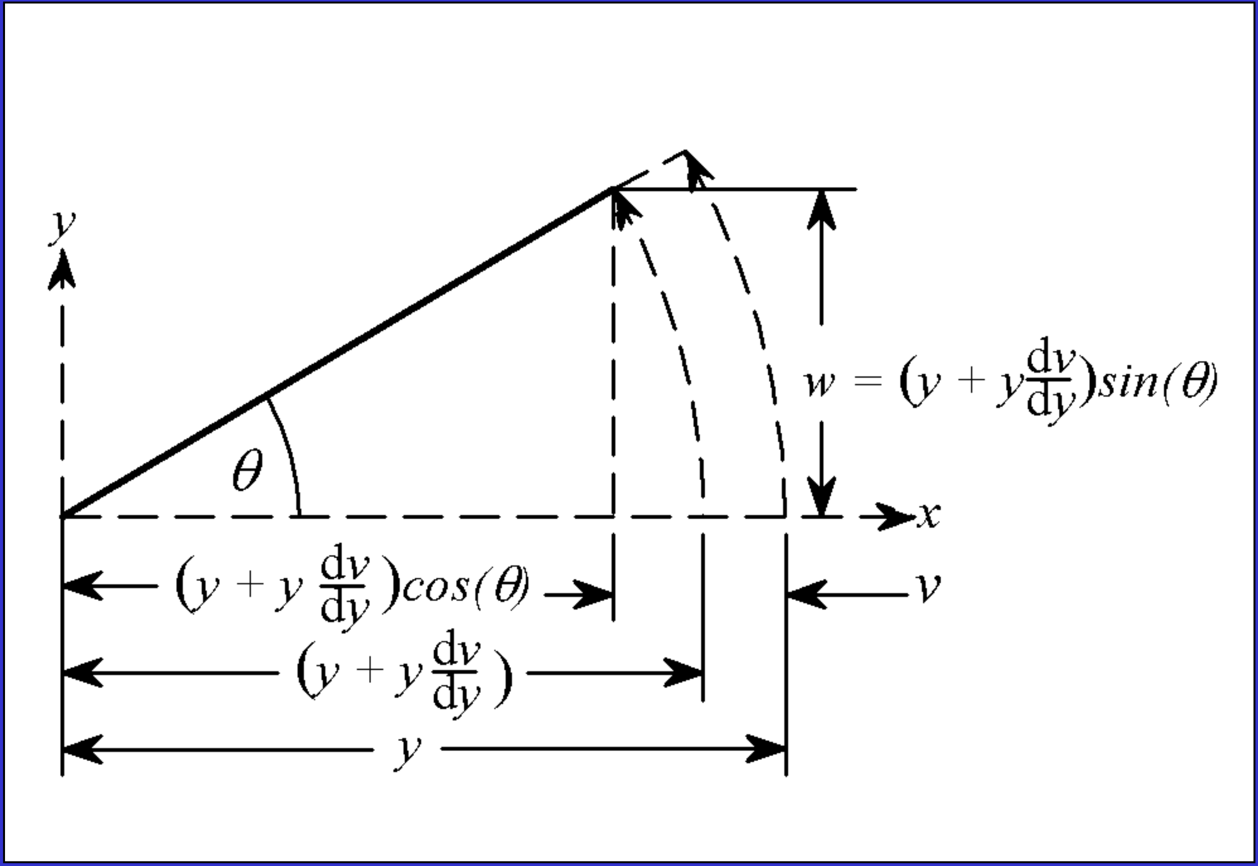
- *v* at the upstream boundary is then,

$$v = y\sqrt{1-2\varepsilon_o}\cos(\theta) - y$$

- θ is the angle of twist.
- *w* is made a function of *v* and θ to produce the twist.

$$w = y\sqrt{1-2\varepsilon_o}\sin(\theta)$$

Upstream condition for *v* & *w*



Downstream conditions

- The web is held fixed in the z direction. So,

$$w = 0$$

- In the y direction the normal entry rule is used.
  - *in a steady state, the path of a particle in the web aligns itself with the direction of the roller surface velocity (normal to the axis of the roller) .*
- The direction of a particle path is defined by the direction cosines.

Downstream conditions

- Referring to the table of direction cosines, the particle path vector is,

$$\vec{P} = \left(\frac{1+u_x}{\sqrt{1+2\varepsilon_{xx}}} \right) \vec{i}_1 + \left(\frac{v_x}{\sqrt{1+2\varepsilon_{xx}}} \right) \vec{i}_2$$

- The tangent of the angle of entry is therefore,

$$\tan(\psi) = \frac{v_x}{1+u_x} = 0 \quad or \quad v_x = 0$$

- Interesting to note this is same result as in 2005 paper, using a completely different method.

Downstream conditions

- In the x direction the normal strain rule is used.
  - *In a steady state, the ratio of the stretched lengths of an infinitesimal patch of the web at two successive rollers is proportional to the respective ratios of the web velocities at the two rollers. This can be stated mathematically as,*

$$\frac{1+\varepsilon_{xu}}{1+\varepsilon_{xd}}=\frac{V_u}{V_d}$$

  - V_d is the downstream velocity, V_u is the upstream velocity, ε_{xu} is the upstream strain and ε_{xd} is the downstream strain.

Downstream conditions

- So, the condition in the x direction becomes

$$\varepsilon_{xx} = \frac{V_u}{V_d}\left(1+\varepsilon_{xo}\right)-1$$

- ε_{xx} is the downstream strain normal to the axis of the roller, ε_{xo} is the strain at the entrance to the upstream roller, V_d is the downstream surface velocity and V_u is the upstream surface velocity.

Edge conditions

- At the edges, both the normal and tangential stresses are zero.
- In each case the total stress projected onto the y axis must be used (the terms inside the brackets of the y derivatives in the equation of equilibrium). So,

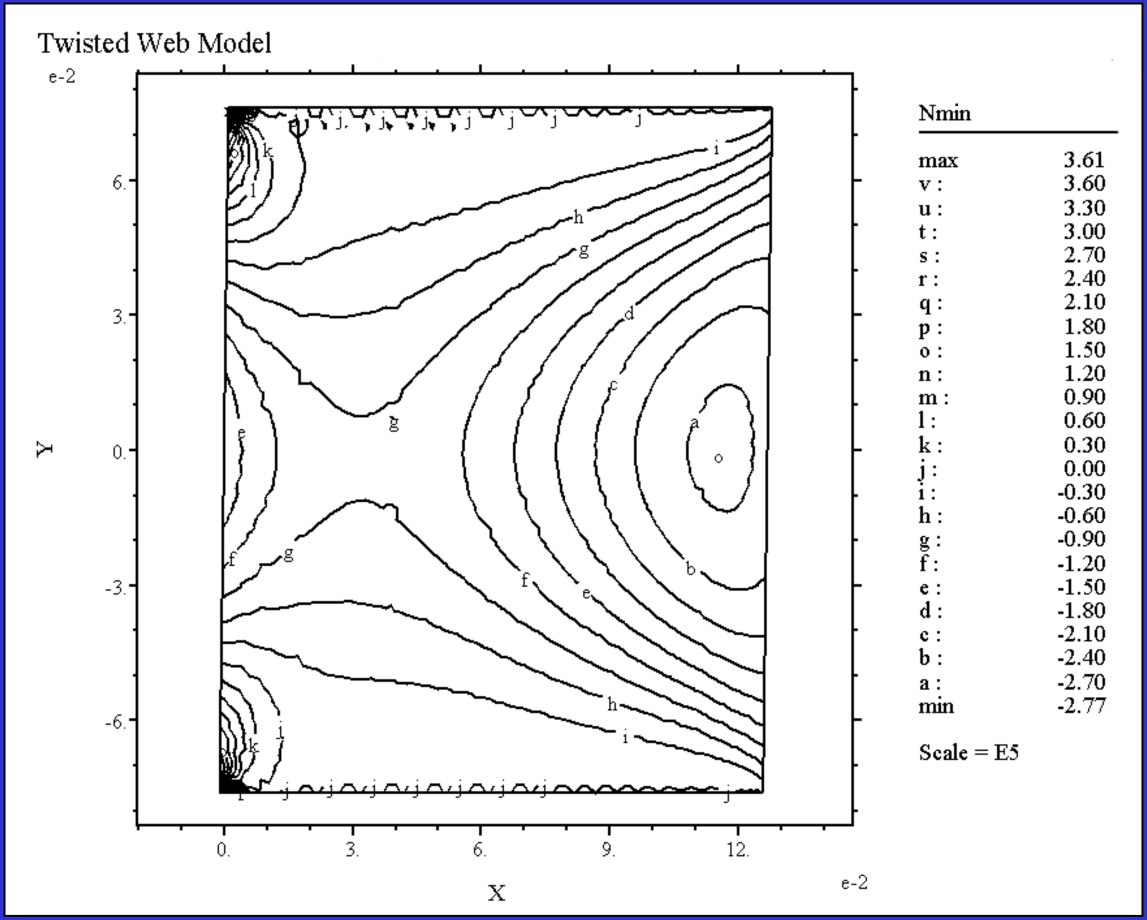
$$\left(1+u_x\right)\sigma_{xy}+u_y\sigma_y=0$$

$$\left(1+v_y\right)\sigma_{yy}+v_x\sigma_{xy}=0$$

Edge conditions

- There is no y derivative in the model equation that defines *w* .
- This implies that the y variation in *w* is completely determined by the x derivative and the conditions at the other boundaries.
- In other words, the boundary conditions on *w* at the rollers define a surface to which the membrane conforms.
- It is possible with FlexPDE to specify this requirement and allow it to solve for the surface.

Typical Results 1



Principal minimum stress (CD)

PET web

Twist = 5 degrees

Length = 0.108 m

Width = 0.152 m

Thickness = 23.4e-6 m

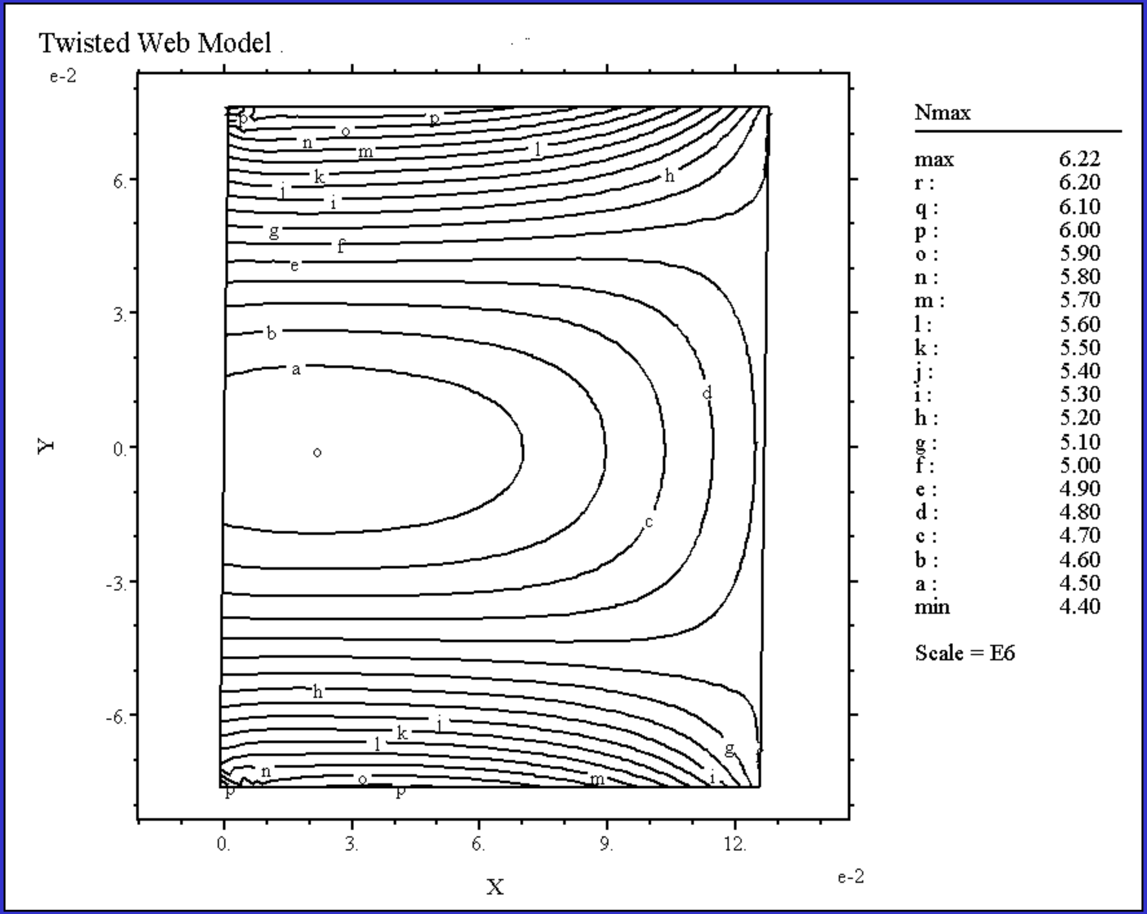
Modulus = 4.13e9 Pa

Tension = 26.7 N,

μ = 0.3,

Diameter = 0.0736 m.

Results 2

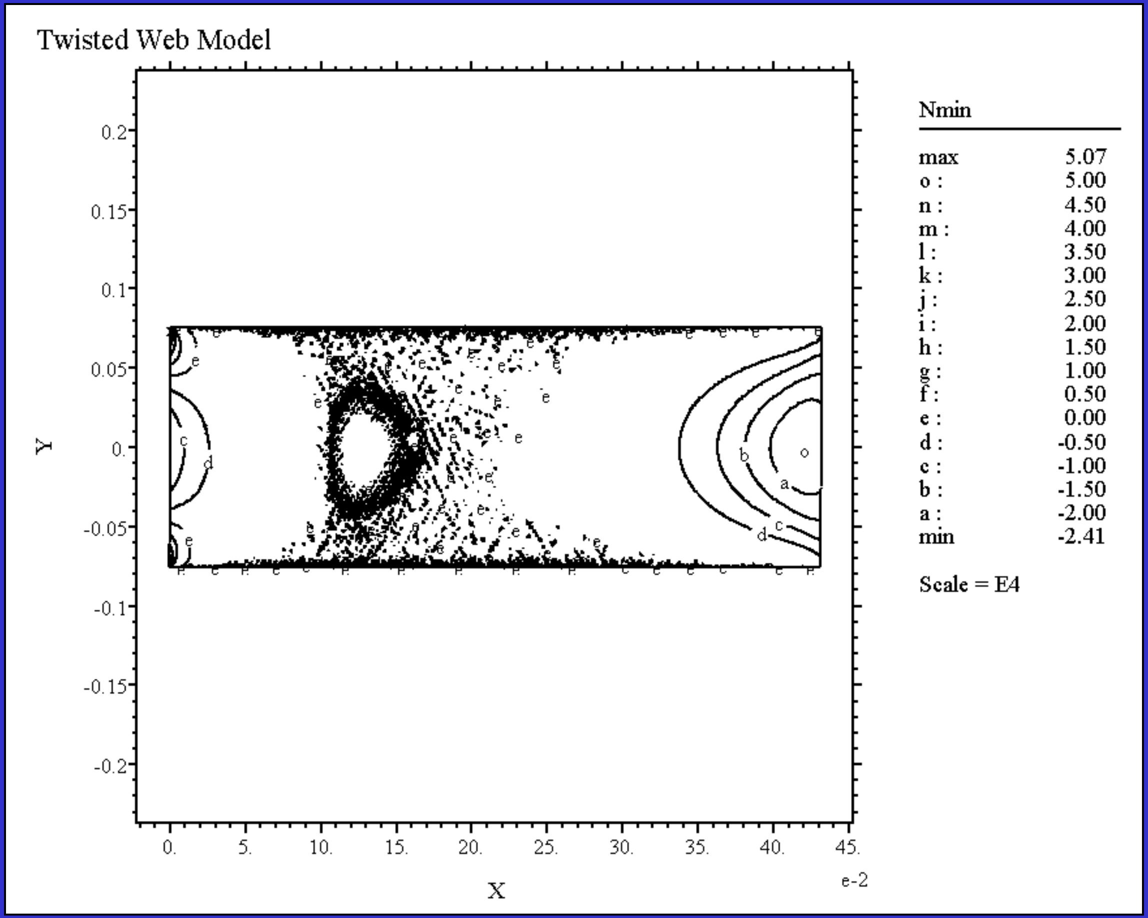


Principal maximum stress

(MD)

Same web as in previous slide

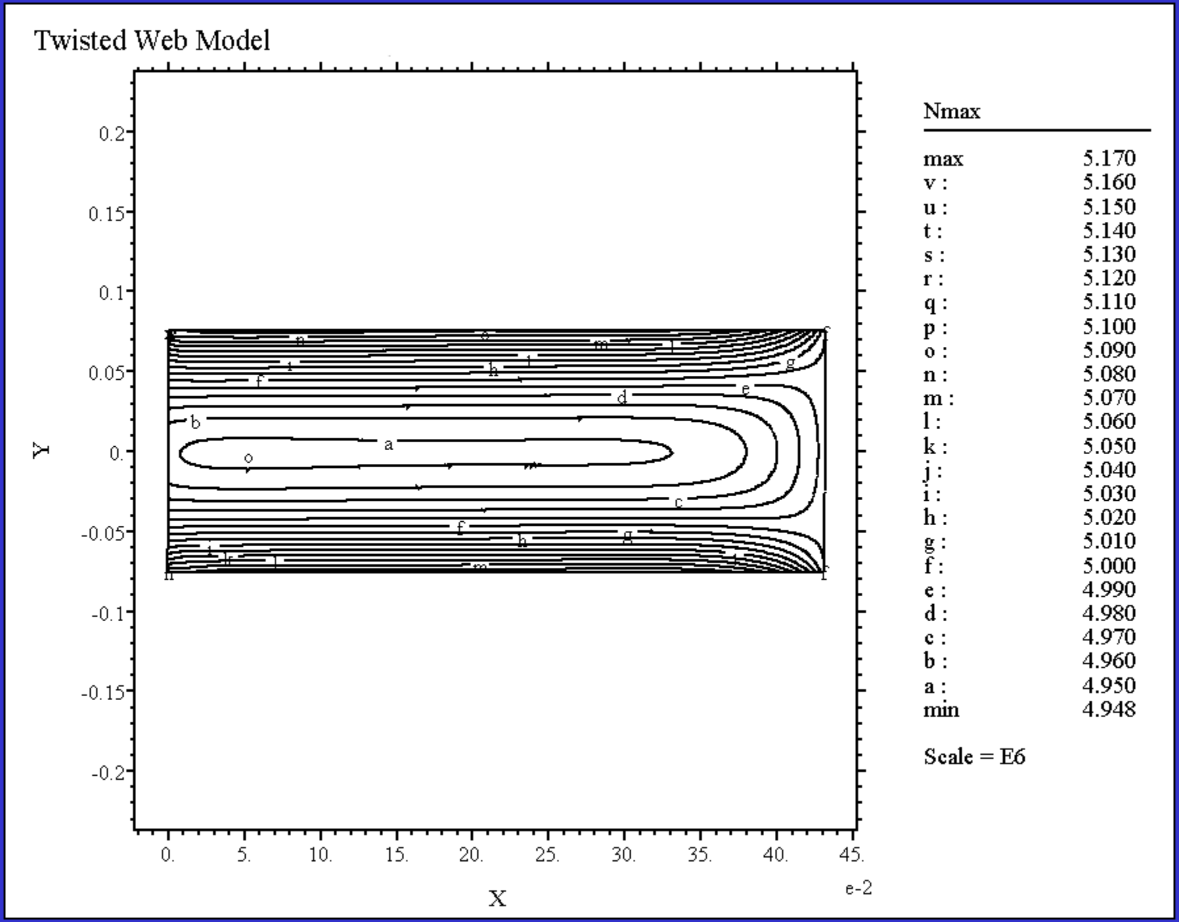
Results 3



CD Stress for same web as before except

L = 0.432 m

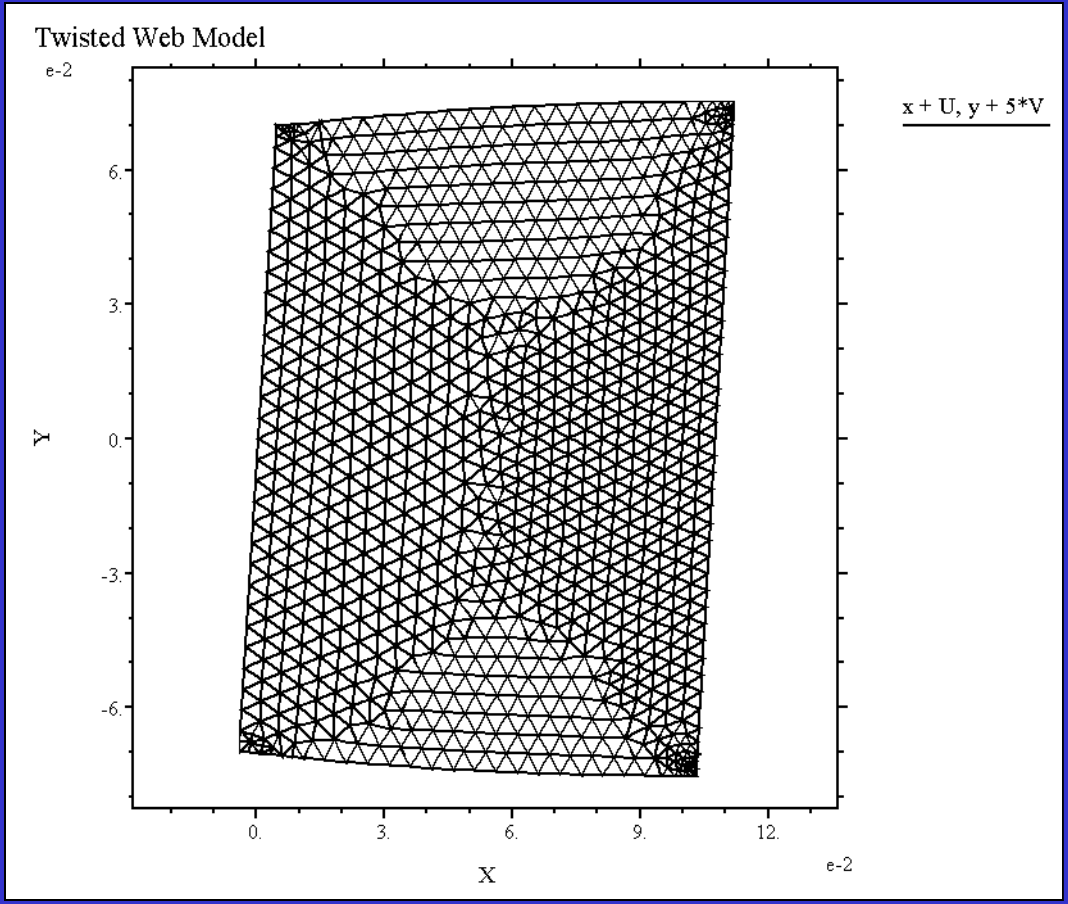
Results 4



MD Stress

L = 0.432 m

Hypothesis to explain compressive stress



Projection of deformed boundaries on x-y plane.

Checked by eliminating normal entry in two ways.

Comparison with experiment

- Good and Straughan performed a series of excellent experiments in which they increased the angle of twist until wrinkles occurred.
- Developed a model (G-S) based on parabolic shaped MD stress profile applied to the ends.
- Used Timoshenko's critical stress, σ_{cr}, for buckling of a cylinder to predict the angle of wrinkling, ϕ_{cr}

Comparison with experiment 1

	L (m)	σ_{xx} (Mpa)	ϕ_{cr} (Deg.)	σ_{cr} (Mpa)	σ_{yG} G-S Flat (Mpa)	σ_{yF} FEA Twisted (Mpa)	σ_{yG}/σ_{yF}	ϕ_{cr}/L (Deg./m)
1	0.108	15	4.1	-1.59	-1.85	-0.86	2.2	38
2	0.108	30	4.6	-1.59	-2.33	-1.1	2.1	43
3	0.464	15	10	-1.59	-2.0	-0.27	7.4	22
4	0.464	30	9.8	-1.59	-1.92	-0.27	7.0	21

PET

Width = 0.152 m, Thickness = 23.4e-6 m,

Modulus = 4.13e9 Pa, μ = 0.3,

Roller diameter = 0.0736 m, Uncoated aluminum surface

Comparison with experiment 2

	L (m)	σ_{xx} (Mpa)	ϕ_{cr} (Deg.)	σ_{cr} (Mpa)	σ_{yG} G-S Flat (Mpa)	σ_{yF} FEA Twisted (Mpa)	σ_{yG}/σ_{yF}	ϕ_{cr}/L (Deg/m)
5	0.127	5	2.7	-1.21	-0.76	-0.28	2.7	21.3
6	0.127	15	3.0	-1.21	-0.94	-0.35	2.7	23.6
7	0.127	25	3.0	-1.21	-0.94	-0.35	2.7	23.6
8	0.432	5	7.4	-1.21	-1.29	-0.18	7.2	17.3
9	0.432	15	8.8	-1.21	-1.82	-0.26	7.0	20.4
10	0.432	25	9.0	-1.21	-1.90	-0.28	6.8	20.8
11	0.584	5	10.8	-1.21	-1.28	-0.21	6.1	18.5
12	0.584	15	11.8	-1.21	-1.53	-0.25	6.1	20.2
13	0.584	25	12	-1.21	-1.59	-0.27	5.9	20.5
	0.584	5	10.5	-1.21	-1.21			

PET

Width = 0.152 m, Thickness = 17.8e-6 m,

Modulus = 4.13e9 Pa, μ = 0.3,

Roller diameter = 0.0736 m, High friction coating

Comparison with experiment 3

	L (m)	σ_{xx} (Mpa)	ϕ_{cr} (Deg.)	σ_{cr} (Mpa)	σ_{yG} G-S Flat (Mpa)	σ_{yF} FEA Twisted (Mpa)	σ_{yG}/σ_{yF}	ϕ_{cr}/L (Deg./m)
14	0.127	13	1.7	-0.96	-0.65	-0.22	3.0	13.5
15	0.127	26	2.1	-0.96	-0.99	-0.25	4.0	16.5
16	0.127	40	2.7	-0.96	-1.63	-0.58	2.8	20.5
17	0.584	13	6.5	-0.96	-1.00	-0.16	6.3	11.1
18	0.584	26	7.8	-0.96	-1.44	-0.23	6.3	13.4
19	0.584	40	7.9	-0.96	-1.48	-0.24	6.2	13.5

PEN

Width = 0.152 m,
Thickness = 6.6e-6 m,
Modulus = 8.87e9 Pa,
μ = 0.3,
Roller diameter = 0.0736 m,
High friction coating

Comparison with experiment 4

	L (m)	σ_{xx} (Mpa)	ϕ_{cr} (Deg.)	σ_{cr} (Mpa)	σ_{yG} G-S Flat (Mpa)	σ_{yF} FEA Twisted (Mpa)	σ_{yG}/σ_{yF}	ϕ_{cr}/L (Deg./m)
20	0.127	3.9	3.5	-0.28	-0.11	-0.036	3.1	27.6
21	0.127	13.1	5.0	-0.28	-0.21	-0.076	2.8	39.4
22	0.584	3.9	13.5	-0.28	-0.17	-0.025	6.8	19.7
23	0.584	13.1	15.0	-0.28	-0.20	-0.032	6.3	22.4

Polyethylene

Width = 0.152 m, Thickness = 50.8e-6 m

Modulus = 0.34e9 Pa, μ = 0.3,

Roller diameter = 0.0736 m, High friction coating

Conclusions 1

- The FEA model shows significantly lower levels of CD compressive stress at the critical angle than the G-S model. They are 1/3 to 1/7 as large.
- With the exceptions of tests 1, 2 and 16, the FEA model produced CD stresses that varied less with tension and span length.

Conclusions 2

- The ratio of the stress magnitudes between the two models changes with span length. For a given length, the ratio is approximately the same across all the tests. This may be due to the difference in the behavior of the MD stress profiles in the two models. The FEA model tends to have a parabolic stress profile throughout its length whereas the G-S model does not. The difference becomes more significant for long spans.

Conclusions 3

- Both models show very little change in CD stress levels with tension.
- There is currently nothing in the FEA modeling technique to detect CD elastic instability. More work should be done to incorporate the features of the von Karman equations.

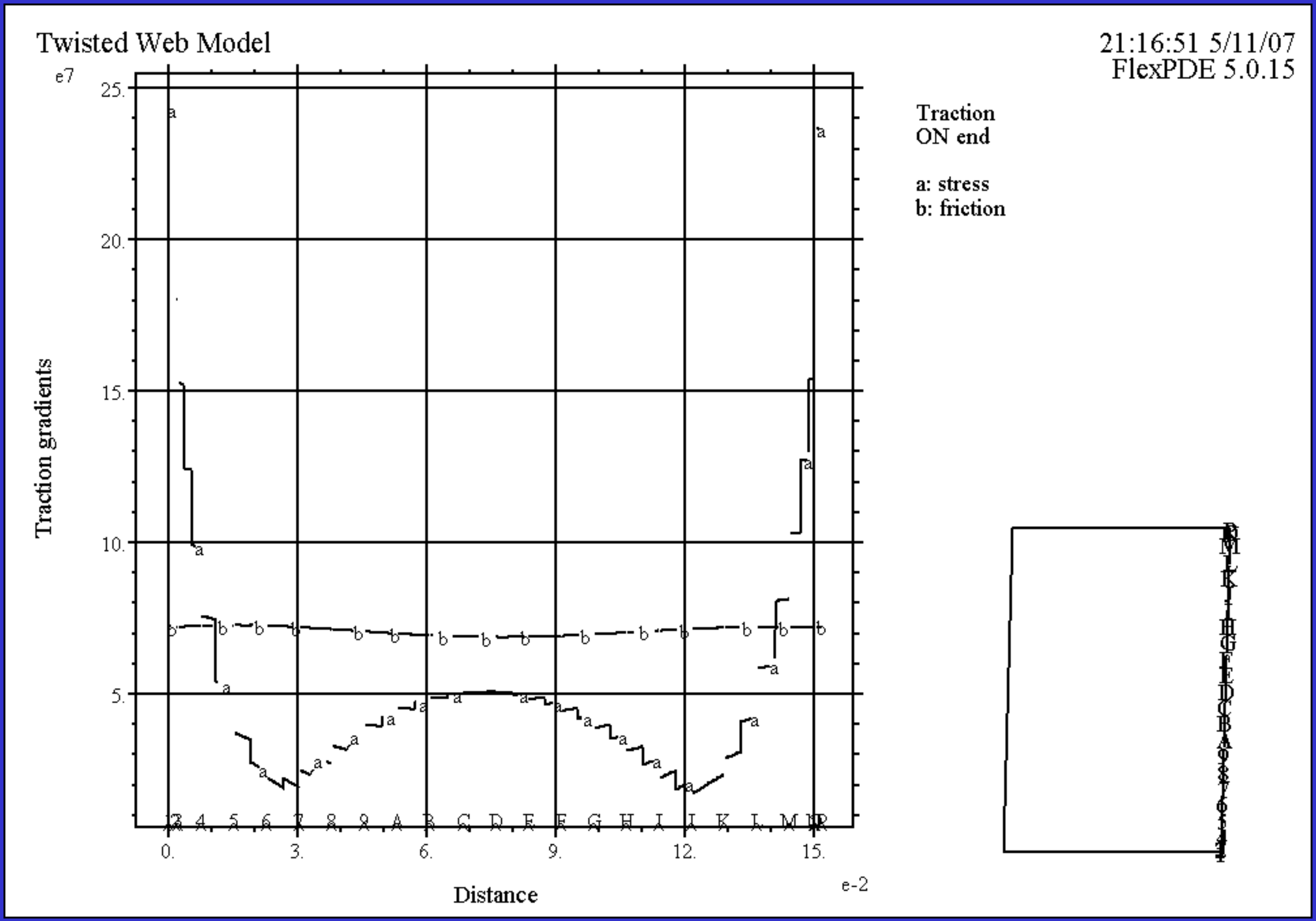
Conclusions 4

- It should be emphasized that the G-S model is still the best tool available for predicting the onset of wrinkling with twist. It may overestimate the magnitude of the CD stress. But if that is the case, the buckling criterion must be doing the same thing because the results agree well with experiment.

Slipping 1

- Twist angle changed very little with tension in the Good-Straughan tests, except in a few cases where both friction and tension were low.
- Good and Straughan suggested that low friction between the web and roller might have allowed troughs to flatten at the roller. This conclusion is supported by the graph in the next slide, which shows a plot of the magnitude of the stress and friction rates at each point across the web.
- It is based on unpublished work by the author.

Slipping 2



Twisted Web Model
21:16:51 5/11/07
FlexPDE 5.0.15
Traction
ON end
a: stress
b: friction
Traction gradients
e7
25.
20.
15.
10.
5.
0.
3.
6.
9.
12.
15.
e-2
Distance

Thank you