A New Method for Analyzing the Deformation and Lateral Translation of a Moving Web

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Overview

It's based on:

- Nonlinear elasticity theory (sometimes called finite deformation theory).
- Two boundary conditions for the downstream roller. One is an extended version of the normal entry rule and the other is new.
- Use of a general-purpose FEA partial differential equation solver to produce numerical solutions for stress and deformation fields throughout spans.

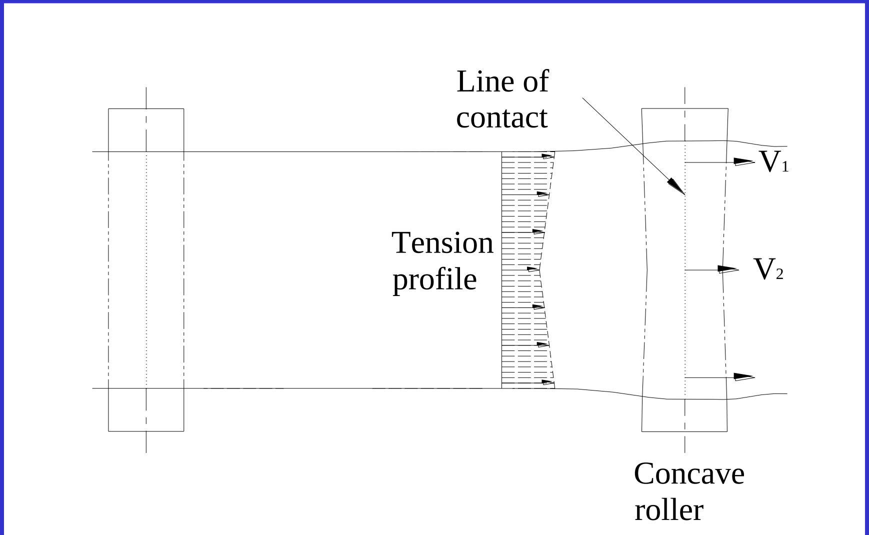
The Usual Assumptions Are Made.

- Traction locks the web to the roller at the entry point so that the span is isolated from changes downstream.
- The web slips loose in a short zone at the exit of a roller as it comes under the influence of the next span.
- The turning torque of the rollers is negligible.
- Viscoelastic and inertial effects are neglected.

Something's Been Missing

- There is the normal entry rule which is an accepted general principle.
- But, one more general rule seems to be needed.
- There is a clue in the fact that experienced web handling practitioners often resort to the following qualitative idea.

Mass Flow



Spreading is sometimes explained this way.

Higher speed at the edges than the center changes the stress distribution in such a way that the normal entry rule can be satisfied only if the web spreads laterally.

This concept has been used for many years [Notably by Feiertag in his semiannual WHRC seminar] to explain the spreading action of tape bumpers. But, it has never been incorporated into a comprehensive, quantitative model.

Earlier work

- Shelton recognized the role of conservation of mass in his 1968 dissertation. He used it as an intuitive proof of the zero moment condition that he had discovered through experiment.
- Swanson in his 1997 IWEB paper on web spreading devices recognized that mass flow and the normal entry rule interact to produce spreading on a concave roller.
- He used the mass flow concept to estimate the end moment of a beam model for a web on a tapered roller.

Earlier work

- Markum and Good in a 2001 IWEB paper refined Swanson's experiment and developed a similar beam model.
- They used the mass flow concept as Swanson had and developed a similar beam model that predicted the observed separations within 5 to 20%.
- Data from these experiments will be used as a test of the P. D. E. model.

Particle Displacements

Displacement in x direction = u

Displacement in y direction = v

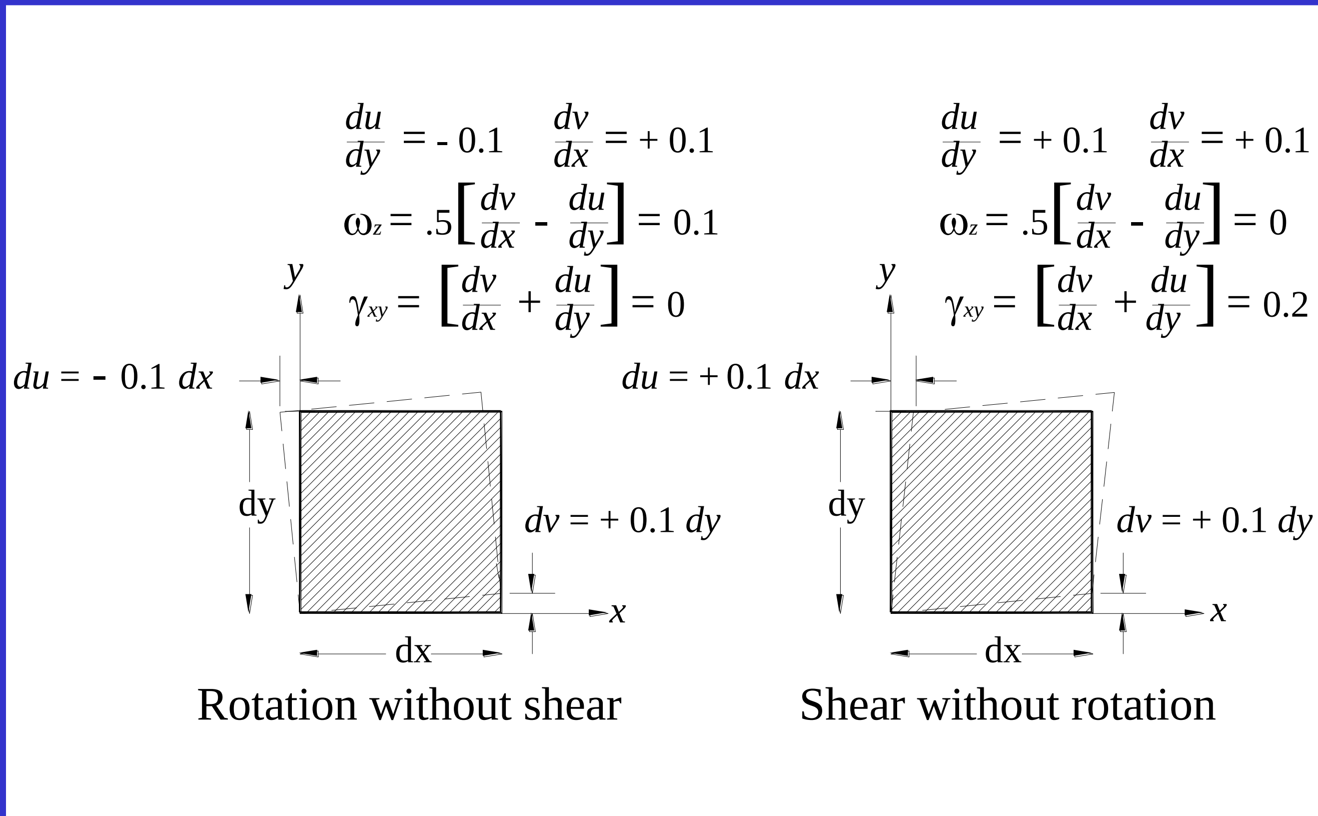
Definitions of Angular Displacements

Angular displacement of *x* coordinate $$v_x = \frac{\partial v}{\partial x}$$
Angular displacement of *y* coordinate $$u_y = \frac{\partial u}{\partial y}$$

Definitions of Strain

Shear	$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$
Rotation	$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

Interpretation of Rotation



Definitions of Strain

- Longitudinal strain $$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\omega_z{}^2 \cong \frac{\partial u}{\partial x}$$
- Cross web strain $$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\omega_z{}^2 \cong \frac{\partial v}{\partial y}$$
- Strain normal to web plane $$\varepsilon_z = \frac{-\mu}{1-\mu}(\varepsilon_x + \varepsilon_y)$$

Deformed Coordinates

Deformed *x* coordinate	$\xi = x + u$
Deformed *y* coordinate	$\eta = y + v$

Definitions of Stress

X-axis	$\sigma_x = \dfrac{E}{1-\mu^2}\left[\varepsilon_x + \mu\varepsilon_y\right]$
Y-axis	$\sigma_y = \dfrac{E}{1-\mu^2}\left[\varepsilon_y + \mu\varepsilon_x\right]$
Shear	$\tau_{xy} = \dfrac{E}{2(1+\mu)}\left[\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right]$

Novoshilov's Equilibrium Equations for Small Rotations

Equilibrium of forces in x direction

$$\frac{\partial}{\partial x}\left[\sigma_x - \omega_z \tau_{xy}\right] + \frac{\partial}{\partial y}\left[\tau_{xy} - \omega_z \sigma_y\right] = 0$$

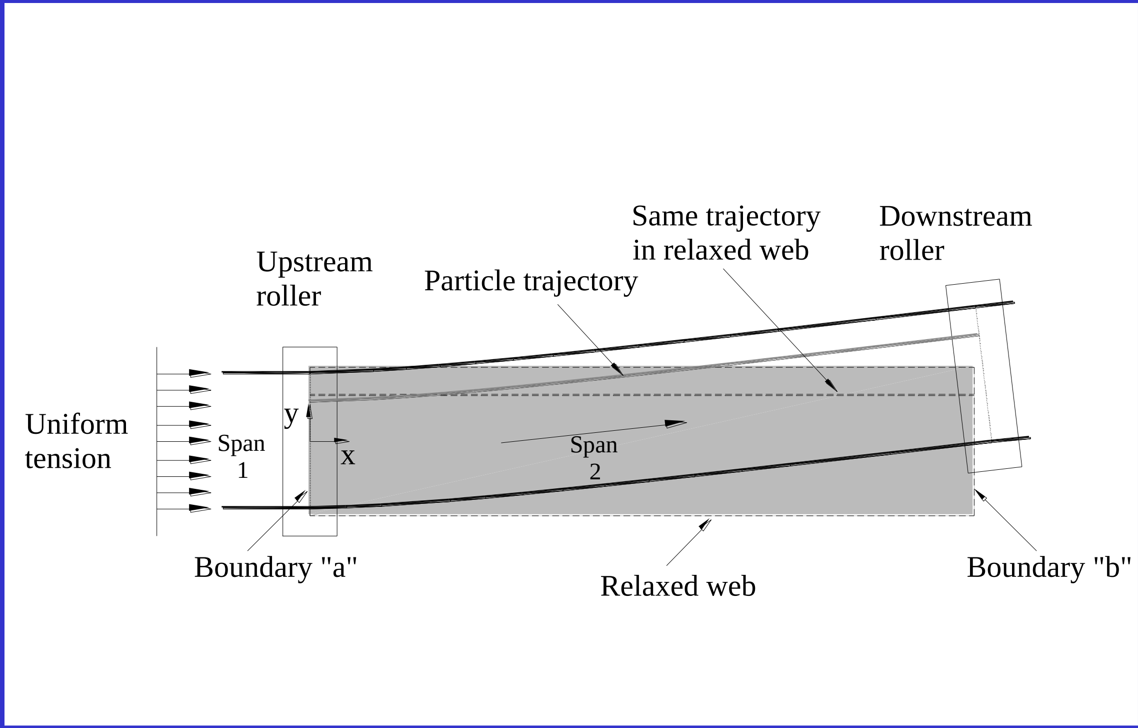
Equilibrium of forces in y direction

$$\frac{\partial}{\partial y}\left[\sigma_y + \omega_z \tau_{xy}\right] + \frac{\partial}{\partial x}\left[\tau_{xy} + \omega_z \sigma_x\right] = 0$$

Conversion From Undeformed to Deformed Coordinates

Deformed x coordinate, ξ $$d\xi = \left(1+\varepsilon_x\right)dx + u_y dy$$
Deformed y coordinate, η $$d\eta = v_x dx + \left(1+\varepsilon_y\right)dy$$

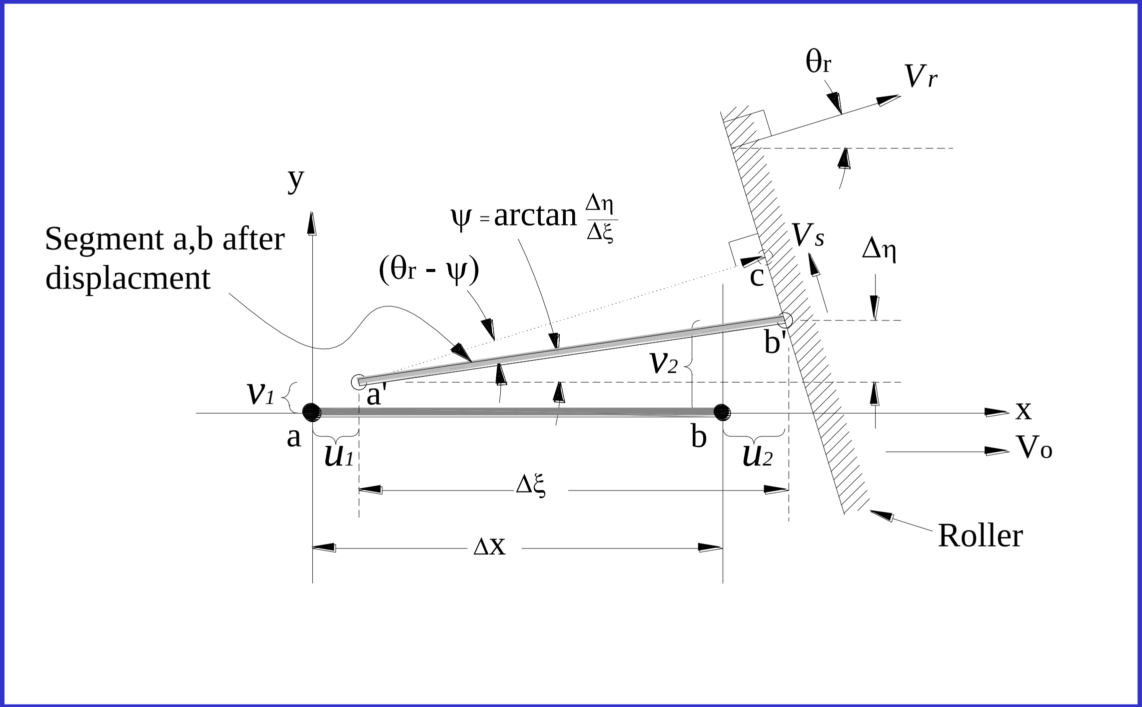
Reference Geometry for Misaligned Roller



Relaxed web aligned with x-axis.

Particle starting at “a” will follow curved trajectory that is straight in the relaxed web.

Normal Entry Rule



$$V_s = V_r \tan\left(\theta_r - \psi\right)$$

In steady state

$V_s = 0$ so $\theta_r = \psi$.

$$\psi = \arctan\left(\frac{\partial\eta}{\partial\xi}\right) \approx \frac{\partial\eta}{\partial\xi}$$

ψ is the angle of a vector tangent to particle path.

Normal Entry Rule

Using the equations that define the deformed coordinates in terms of *x* and *y*,

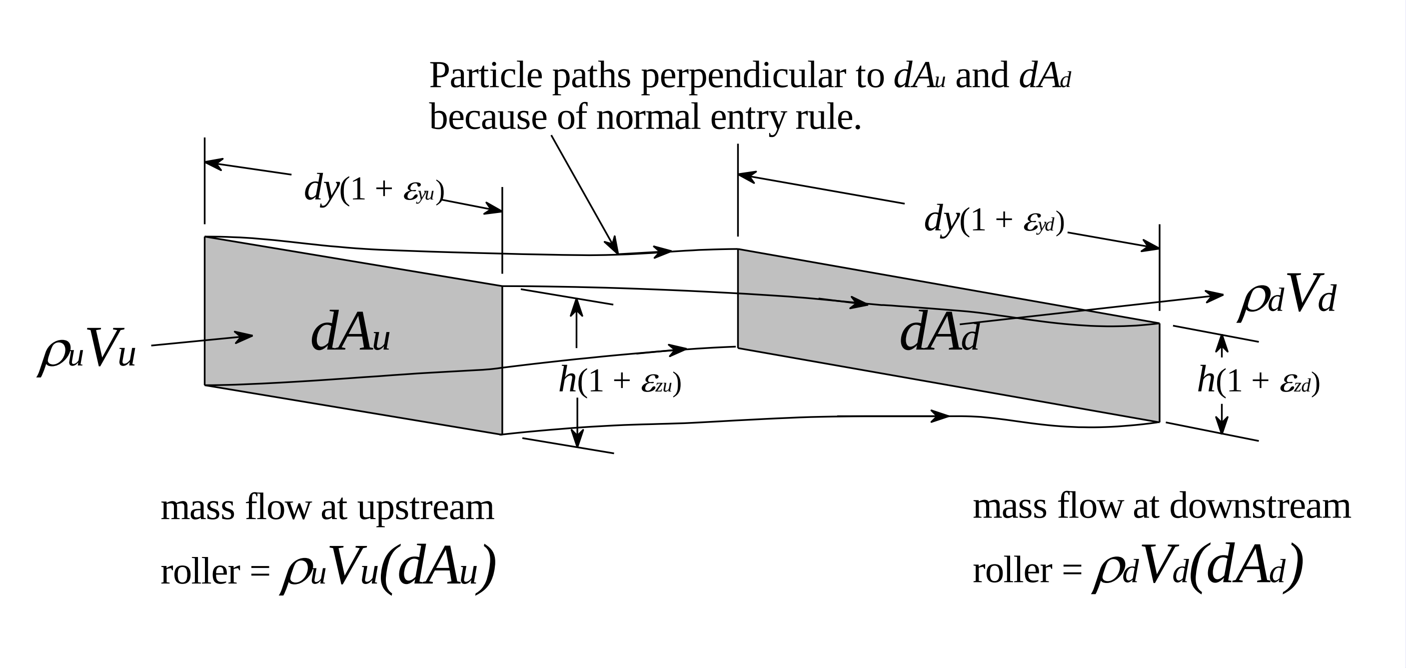
$$\psi = \tan^{-1}\left\{\left[v_x dx + \left(1+\varepsilon_y\right)dy\right]\left[\left(1+\varepsilon_x\right)dx + u_y dy\right]^{-1}\right\} = \theta_r$$

For uniform web *y* = constant and *dy* = 0

So,

$$\tan^{-1} v_x (1+\varepsilon_x)^{-1} \approx v_x = \theta_r$$

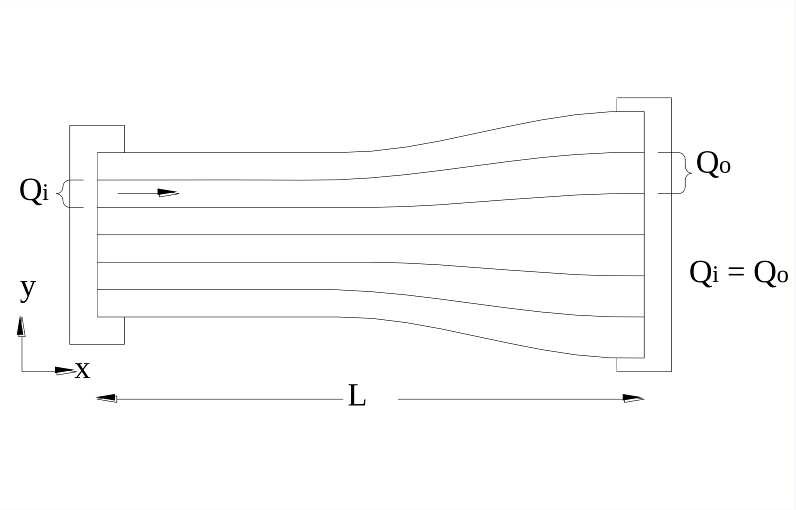
Normal Strain Rule



Start by considering the mass flow at upstream and downstream rollers through increments of width that were equal to *dy* in the relaxed web. The mass flows must be equal. However,

$\rho_u \neq \rho_d$ and $dA_u \neq dA_d$.

Normal Strain Rule



$$Q_i = \frac{dm}{dt} = V_u dy\left(1+\varepsilon_{yu}\right)h\left(1+\varepsilon_{zu}\right)\rho_u$$

$$\rho_u = \frac{\rho_o}{\left(1+\varepsilon_{xu}\right)\left(1+\varepsilon_{yu}\right)\left(1+\varepsilon_{zu}\right)}$$

$$Q_i = V_u\left(dy\right)\left(h\right)\left(\rho_o\right)\left(1+\varepsilon_{xu}\right)^{-1}$$

V_u = Upstream velocity

$$Q_o = V_d\left(dy\right)\left(h\right)\left(\rho_o\right)\left(1+\varepsilon_{xd}\right)^{-1}$$

V_d = Downstream velocity

Normal Strain Rule

Equating Q_o to Q_i and solving for ε_{xd} ,

$$\varepsilon_{xd} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = \frac{V_d}{V_u}\left(1+\varepsilon_{xu}\right)-1$$

So, at the point of entry onto a roller, the steady state component of strain normal to the roller axis is a function only of the ratio of the circumferential velocities of the rollers at the ends of the span and of the longitudinal strain at the entry to the upstream roller.

And there are no approximations in this relationship other than those underlying elasticity theory.

Normal Strain

- Thus, the unremarkable fact that the mass of a piece of web doesn't change when it's stressed leads to a boundary condition that seems quite surprising when viewed from the standpoint of elasticity theory.
- The other strains, ε_y and ε_z also change. But, they factor out of the relationship because of their effect on density.

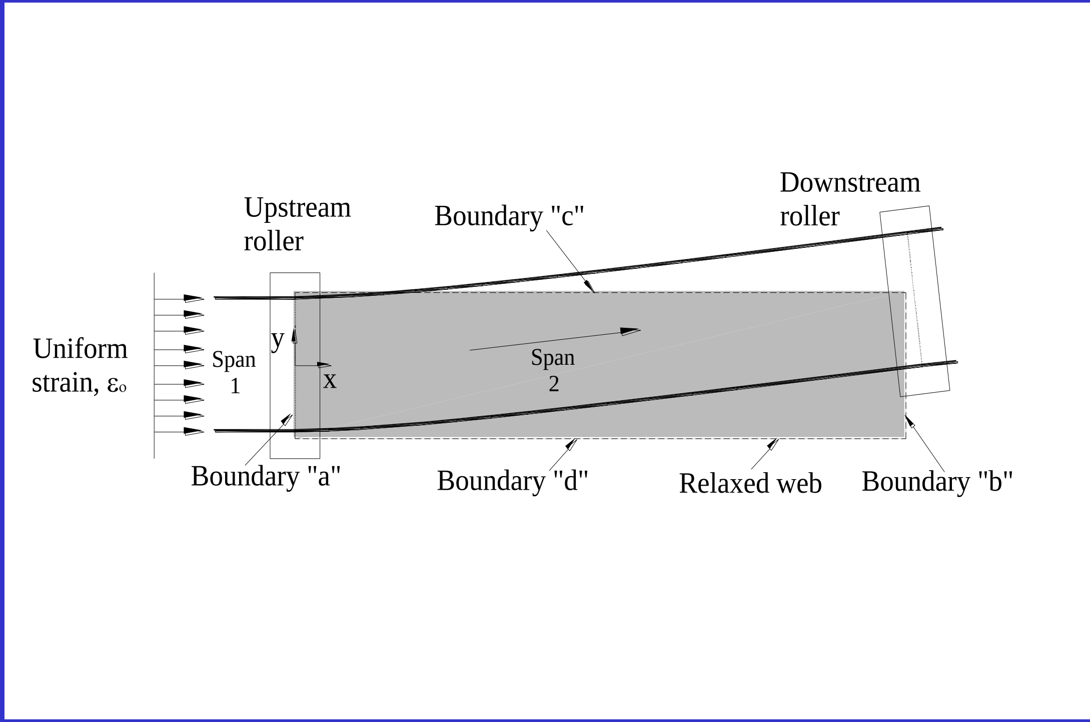
Profile Displacement

- When the web is displaced laterally on a nonuniform roller, the displacement must be taken into account when determining the effect of the profile.
- The problem may be eliminated in a P. D. E. solver by using $(y + v)$ to define V_d in the normal strain relation. When this is not possible, a recursive calculation may be used in which the displacement caused by the profile is used to recalculate its relative position.

Application of the Boundary Conditions to a Misaligned Roller

The first test of the P. D. E. Model is a comparison with Shelton's 1968 dissertation which provided experimental verification of a very effective beam theory model.

Boundary Conditions



At “c” and “d”

$$\sigma_n = 0 \qquad \tau_n = 0$$

At “a”

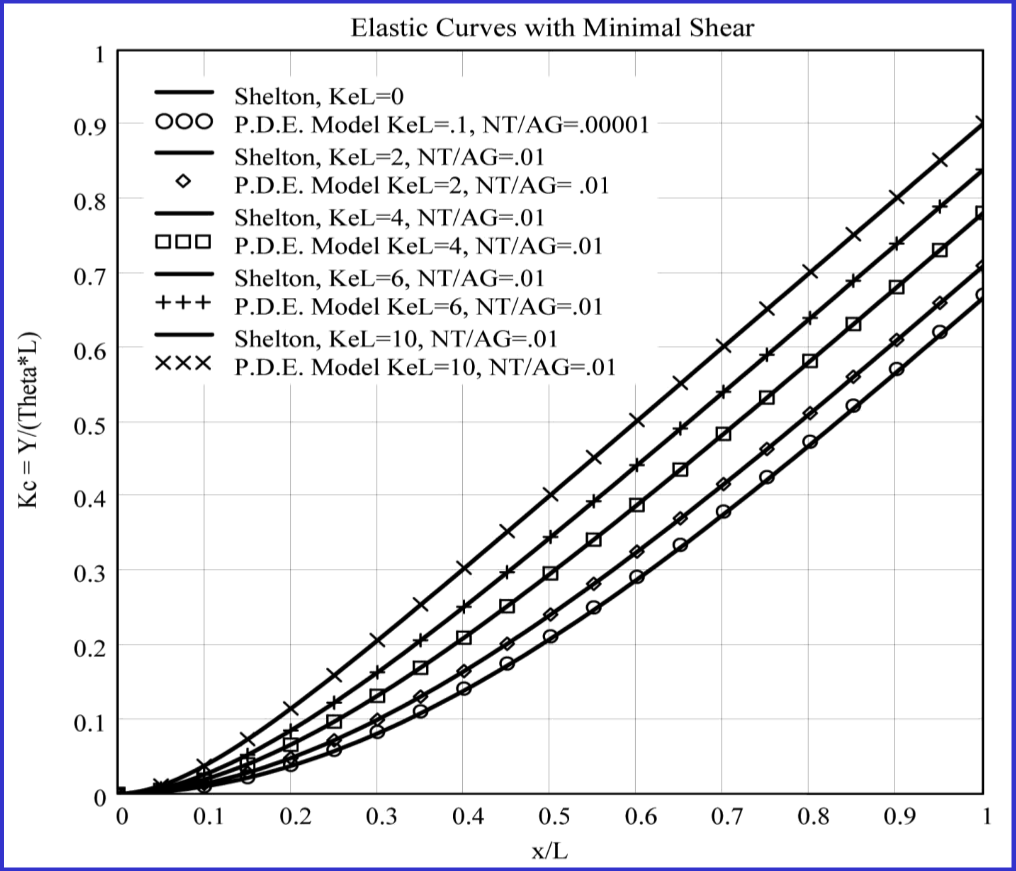
$$u = 0 \qquad v = -\mu y \varepsilon_o$$

At “b”

$$v_x = \theta_r$$

$$\varepsilon_x = 1 - \frac{V_u}{V_d}\left(1 - \varepsilon_o\right)$$

Elastic Curves for Long Spans



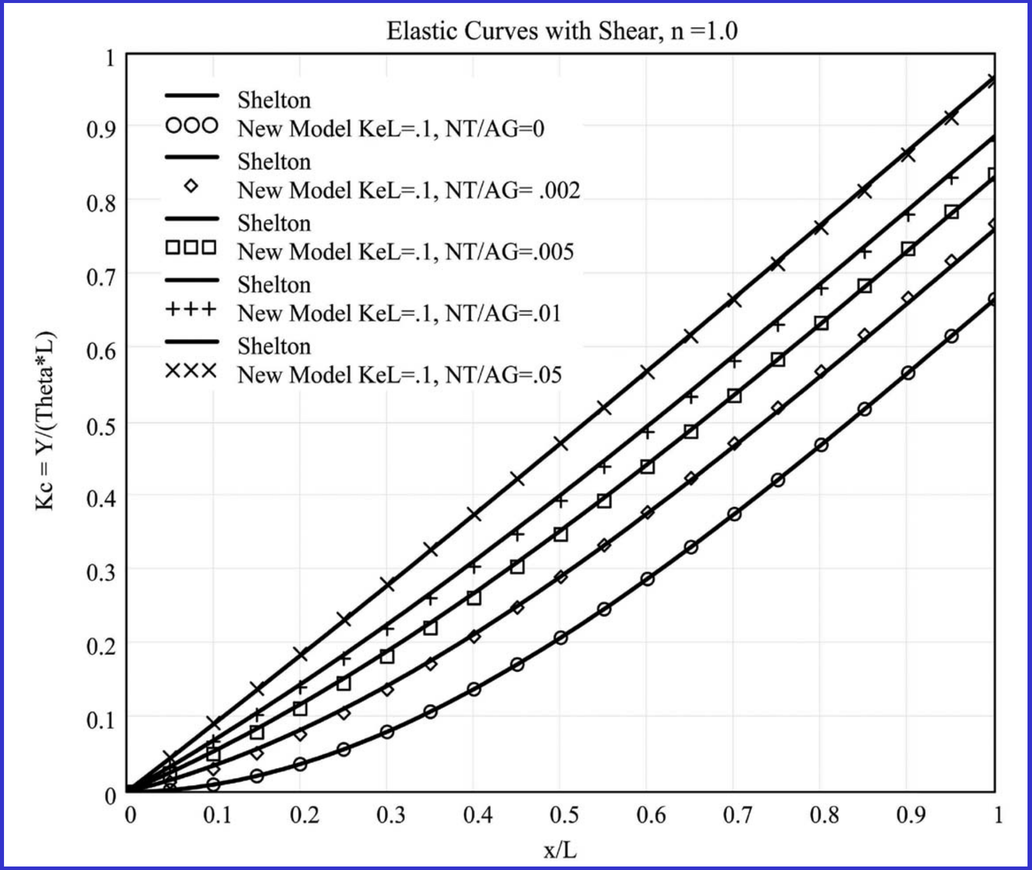
Solid lines – Shelton

Data points – P. D. E.

At KeL = 0 the inputs are not exactly the same.

Without nonlinear terms, P. D. E. results would all look like curve for KeL = 0

Elastic Curves for Short Spans

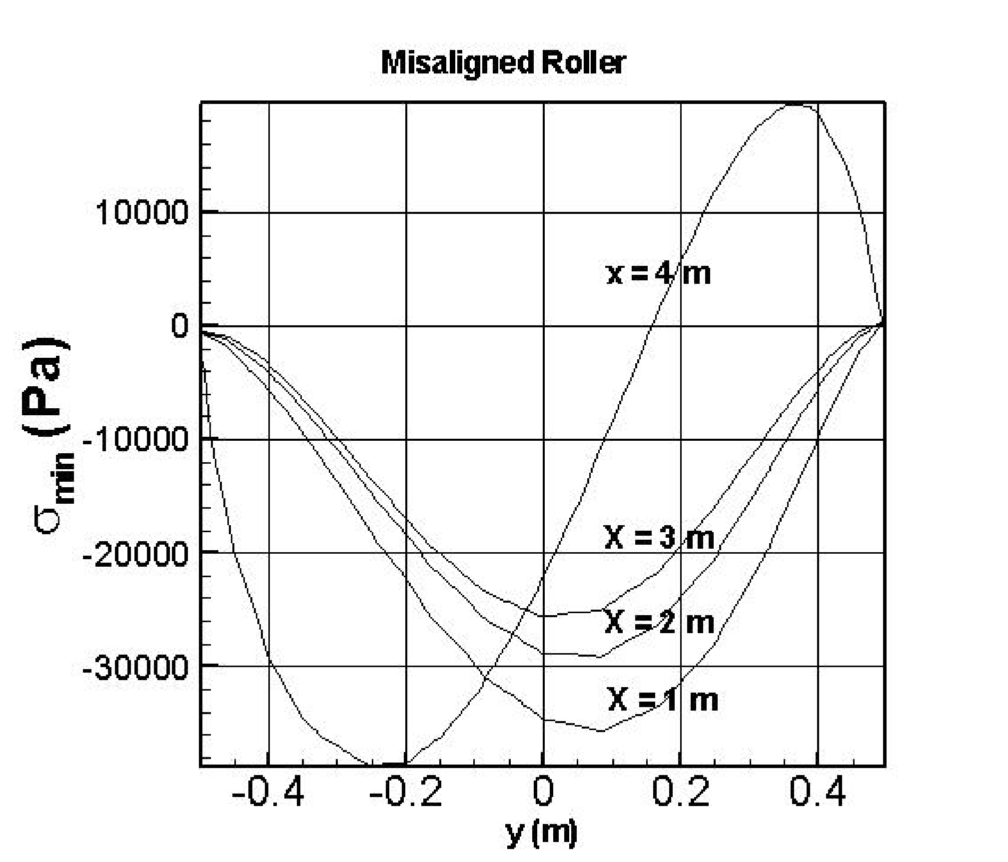


Solid lines – Shelton

Data points – P. D. E.

Example of P. D. E. Model Results

Principal Minimum Stress



L = 4 m

W = 1 m

σ_x = 6.5e6 Pa

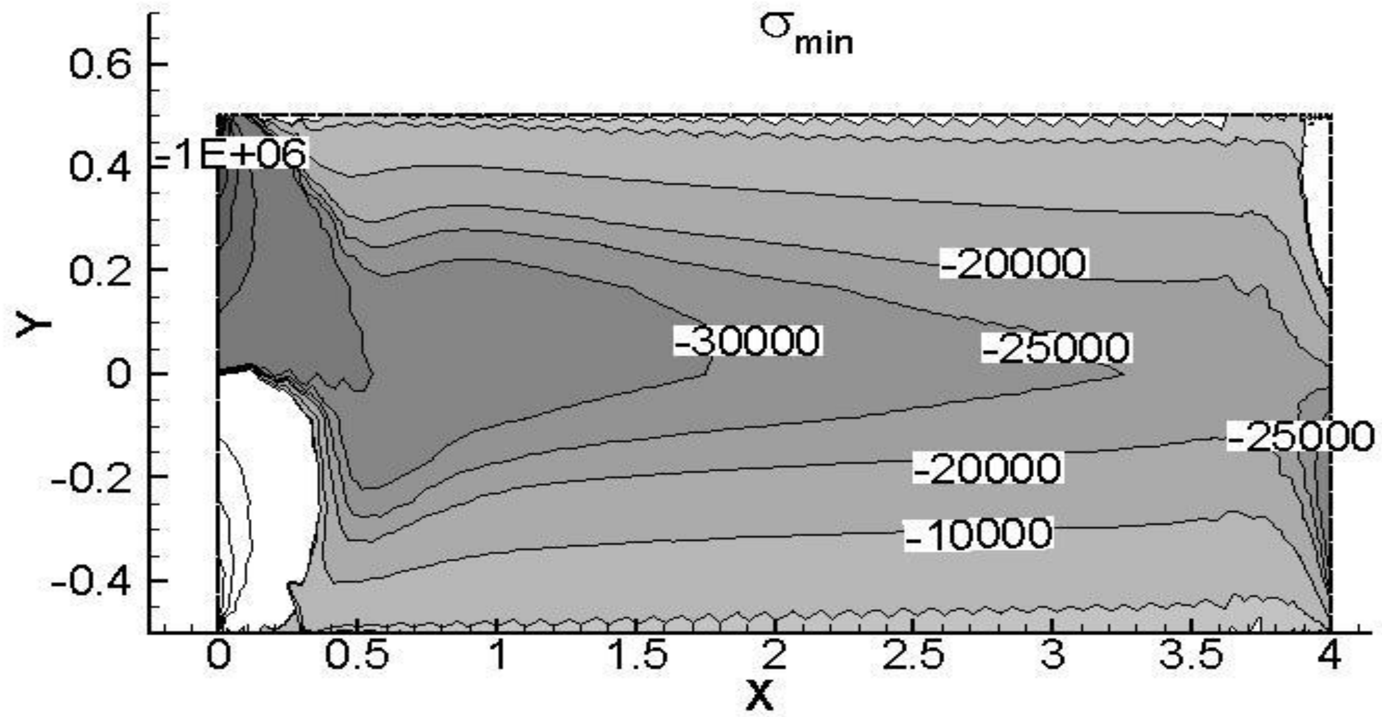
E = 3.1e9 Pa

μ = .35

θ_r = 0.5 deg

Example of Output From P. D. E.

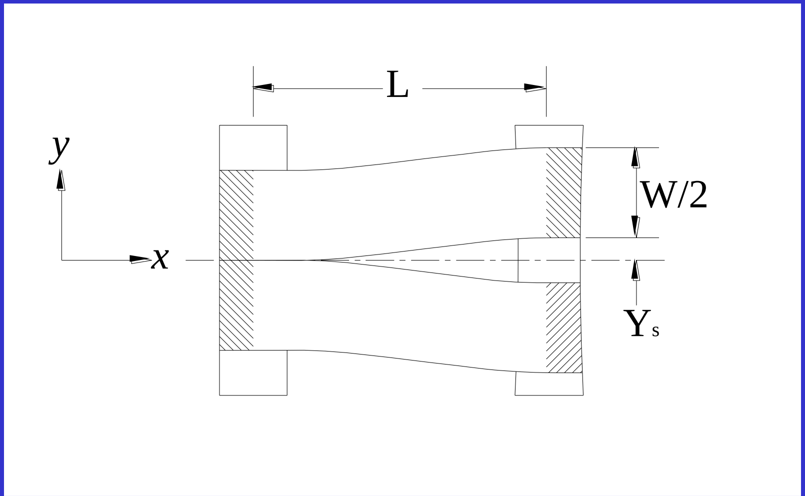
Principal Minimum Stress



Comparison With a Tapered Roller Experiment

- Markum and Good performed an experimental evaluation of a concave roller by splitting a web as it exited an upstream roller and measuring the separation at the spreader. Although it was done to evaluate spreading behavior, it is effectively a tapered roller experiment.
- It's documented well and provides a good test of the P. D. E. model.

Application of Boundary Conditions to a Tapered Roller



At sides:

$$\sigma_n = 0 \qquad \tau_n = 0$$

At downstream roller:

$$v_x = \theta_r \qquad \varepsilon_x = 1 - \frac{V_u}{V_d}\left(1 - \varepsilon_o\right)$$

At upstream roller:

$$v = -\mu \varepsilon_o y \qquad u = 0$$

Tapered Roller Boundary Conditions

For this case, the normal strain rule can be expressed in terms of a function of the fractional difference in the roller Diameters – D_u , upstream and D_d , downstream.

$$f(y) = \frac{D_d(y) - D_u}{D_u}$$

And $V_d = V_u \left[1 + f(y)\right]$ So, $\varepsilon_x = 1 - \frac{1 - \varepsilon_o}{1 + f(y)} \cong \varepsilon_o + f(y)$

Where ε_o is the longitudinal strain from the previous span.

Tapered Roller Boundary Conditions

- Markum and Good's best results were obtained with their "parabolic 2" profile. The concave roller surface was cut with a circular arc of radius $R_o = 10.16m$. The center diameter, D_c was 55.5mm. The diameter as a function of y is approximated mathematically as,

$$D_d\left(y\right) = 2\left(\frac{D_c}{2} + \frac{y^2}{2R_o}\right) == 2\left(0.02776 + 0.0492y^2\right)$$

- This roller had a maximum surface depth of 0.28mm at the center, relative to the location of the web edge.

Comparison of Results

Four sets of results will be presented.

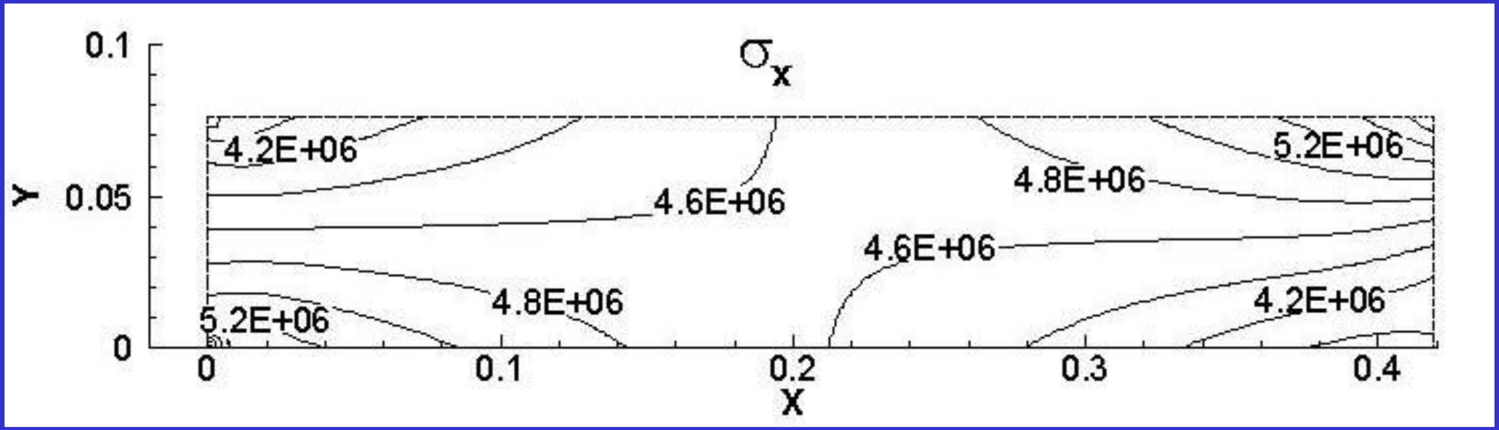
- The first is the experimental data.
- The second is from Markum and Good’s beam model.
- The third is from an enhancement of Markum and Good’s model that uses recursion to compensate for profile displacement.
- The fourth is from the P. D. E. model.

Comparison of Results

#	Material	Modulus (MPa)	Caliper (µm)	Width (m)	Length (m)	Tension (N)	Profile Radius (m)	Profile f(y)
1	LDPE	165.5	25.4	0.152	0.419	8.9	10.16	$.02776 + .0492y^2$
2	"	"	"	"	0.419	17.8	"	"
3	"	"	"	"	0.242	8.9	"	"
4	"	"	"	"	0.242	17.8	"	"

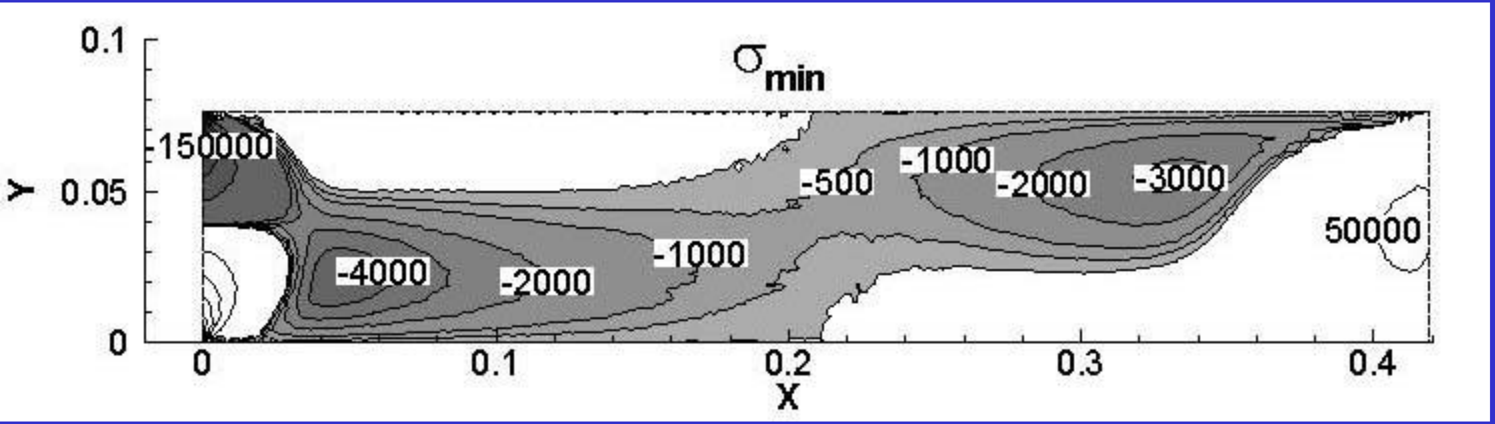
#	Measured Separation 2(Ys) (m)	Beam Model (m)	Pct Error relative to measured	Recursive beam model (m)	Pct Error relative to measured	P. D. E. model (m)	Pct Error relative to measured
1	0.0079	0.0072	-8.30%	0.008	1.33%	0.00792	0.25%
2	0.0075	0.0067	10.4	0.00737	-1.74	0.00702	-6.4
3	0.0031	0.0026	-17.8	0.00264	-14.9	0.00274	-11.6
4	0.0027	0.0024	-10	0.00256	-5.03	0.00255	-5.56

σ_x Field From the P. D. E. Model



There is a nice symmetry in the σ_x field. A plot of the displacement would show that this is because the deformed web has a symmetrical "s" shape. Stresses are in Pascals.

σ_{min} Field From the P. D. E. Model



- The minimum principal stress, σ_{min} , shows that the assumption of $\sigma_y << \sigma_x$, used to estimate the end moment is valid. [σ_{min} is very nearly equal to σ_y because the principal angle is only a few degrees off the x-axis.]
- And as one would expect, there is no compressive stress (shown shaded) near the downstream roller. Stresses are in Pascals.

Conclusions

A new method for solving problems involving deformation and translation of moving webs is now available.

- It is founded on basic elasticity theory in a way that permits the use of general-purpose numerical methods to rapidly solve the partial differential equations.
- It introduces a rigorous definition of the normal entry rule suitable for use with elasticity theory.
- It introduces a new boundary condition for the downstream roller that has the same range of application as the normal entry rule.

Conclusions

- It shows that so long as traction is maintained, the controlling conditions at the entry to a roller are fundamentally geometric with stresses only controlling the relationships between the strains that govern particle paths and mass flow.
- It has been shown to produce solutions that are in agreement with experimental results reported for both misaligned and tapered rollers.
- It is capable of providing detailed descriptions of stress and deformation fields throughout web spans.
- It can be used to identify relationships that help in the creation of simplified models.