A COMPARISON OF MULTI-SPAN LATERAL DYNAMICS MODELS

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ABSTRACT

It's obvious that when a web changes its lateral position at a roller, subsequent spans will be affected. Less obvious, is the fact that transient stress variations will also affect downstream spans. Multi-span models are designed to account for both effects by incorporating a method for transferring the lateral displacements and deformations over rollers. In her 1987 thesis [1] and a follow up paper in the IEEE Transactions [2], Lisa Sievers described three multi-span models.

- Convecting string with zero bending stiffness
- Euler-Bernoulli beam with bending stiffness and no shear
- Timoshenko beam with both bending and shear

Both beam models were capable of modeling a phenomenon called weave regeneration in which an oscillatory lateral disturbance reappears downstream of a web guide which had corrected it. The Timoshenko beam showed better qualitative agreement.

Then, in 1989 Young, Shelton & Kardamilas (YSK) [3] published a description of an Euler-Bernoulli multi-span model and applied it to common roller configurations such as pairs of parallel rollers, displacement guides and steering guides. It transfer's lateral bending deformation across rollers and is functionally identical to the Sievers Euler Bernoulli model. A notable feature of the YSK model is the way it uses transfer functions to interconnect the spans.

The original goal for this paper was to recast the Sievers Timoshenko model into the same analytical form as the YSK model, develop a similar interconnection strategy and then compare the two models quantitatively. A YSK-type Timoshenko model has been developed, and it looks quite plausible. However, it produces a value for the curvature factor that doesn't make sense. After exhaustive troubleshooting, I've concluded that the problem is not due to a procedural error; but is more likely something of a conceptual nature.

As an alternative to the original plan I will 1) describe the problem with the YSK-type Timoshenko model and 2) answer the question, "How much difference does shear

make.” with a quantitative comparison between the Euler-Bernoulli model of the YSK paper and a modified version of the Sievers Timoshenko model. A detailed derivation of the modified Sievers Timoshenko model is described in a companion paper [4] presented at this conference.

NOMENCLATURE

A	cross sectional area of web
a	$1 + nT/AG$
E	elastic modulus
G	shear modulus
h	thickness of web
I	area moment of inertia
J	rotational inertia
K	constant in elastic curve O. D. E.
K_c	curvature constant
L	span length
m	mass per unit length
n	Shear coefficient for Timoshenko beam
r_c	distance from roller to apparent pivot point of bent web
t	time
T	tension
v_o	web velocity in machine direction
v_y	lateral web velocity
x	distance along length of web
y	lateral displacement of web
y_0	lateral web displacement at upstream roller
y_L	lateral web displacement at downstream roller
γ	angle of roller axis
θ	slope of web
ρ	density
ϕ	rotation of cross section
ψ	shear angle
$_0$	subscript indicating value of variable at x = 0
$_L$	subscript indicating value of variable at x = L

In this paper and much of the current literature, variables are referenced to spans. A variable labeled y_{02} would indicate the value of y at $x = 0$ in span 2.

CURVATURE FACTOR

The curvature factor, which will be explained below, is a good indicator of model validity because it relies in an intimate way on the shape of the web when it is controlled by the normal entry rule. Shelton derived and published it for his simplest single span model in 1968 [5] and it shows up in more sophisticated dynamic models as the static gain factor in transfer functions.

At a misaligned roller, a web will “walk” laterally on the roller until it is perpendicular to the roller axis. To do this, it must bend. When viewed from the perspective of the portion of web near the misaligned roller, the web, in its steady state, appears to have pivoted about a point upstream. The effective radius r_c of the pivoting motion is equal to K_cL, where K_c is the curvature factor. For a Timoshenko beam K_c is,

$$K_c = \frac{1}{KL}\frac{\sinh(KL) - KLa\cosh(KL)}{1 - a\cosh(KL)} \qquad \{1\}$$

and,

$$K = \sqrt{\frac{T}{EIa}} \qquad a = 1 + \frac{nT}{AG} \qquad \{2\}$$

T is tension in units of force, E is the elastic modulus, I is the area moment of inertia and G is the shear modulus. For the Timoshenko beam $n = 1.2$. For a Euler Bernoulli beam the same equations are used, but $n = 0$.

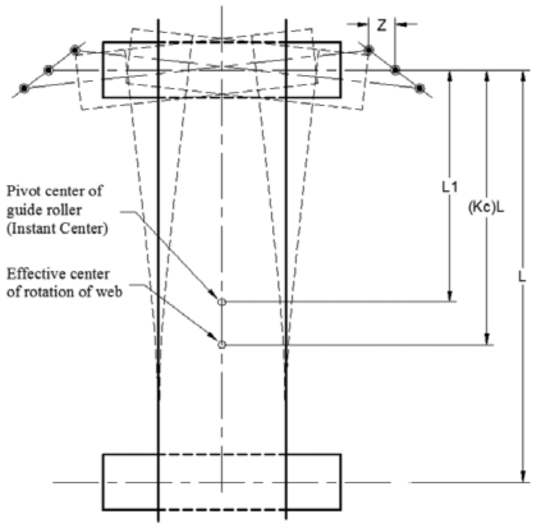

Figure 1
Curvature factor, *Kc*

Figure 2 is a recreation of Figure 2.4.6 in Shelton's dissertation and shows how K_c varies with *KL* and with the shear factor *nT/AG*. It can be seen that K_c ranges from a minimum value of 0.67 to a maximum of 1.0. For large *KL*, K_c approaches 1.0 because all the bending occurs near the upstream roller. For small *KL* with shear, K_c again approaches 1.0 because the web deforms more like a parallelogram with a sharp bend at the upstream roller.

A remote pivot steering guide provides its optimum time response when the pivot center of the guide roller coincides with the center of web rotation.

Four data points are marked on Figure 2, parameters for them are as follows.

L = tabled below	T = 44.5 lbf
h = 0.0034 inches	E = 550,000 psi
W =44.5 inches	nT/AG = 0.0017

L(inches)	*nT/AG*	*KL*	K_c	Symbol
10	0.0017	0.018	0.97	○
22	0.0017	0.040	0.89	□
40	0.0017	0.072	0.80	Δ
120	0.0017	0.216	0.69	◊

Table 1
Parameters for four points on Figure 2

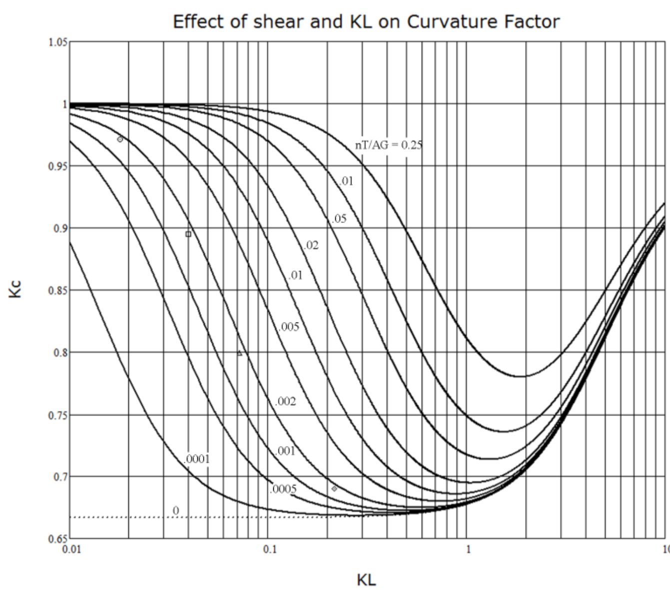

Figure 2
Effect of shear and KL on Kc

SHEAR IN THE TIMOSHENKO MODEL

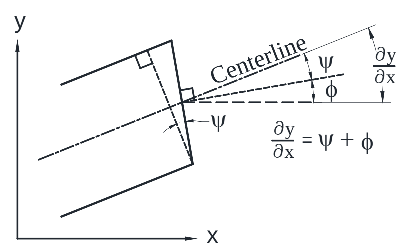

Figure 3
Relationship of Slope, shear and rotation of cross section
$\partial y/\partial x$ = slope , ψ = shear angle , ϕ = angle of cross section rotation

Shear and bending deflections are additive as illustrated in Figure 3. The relationship shown in {3} is fundamental to everything that follows. The total slope is equal to the sum of bending and shear[1].

$$\frac{\partial y}{\partial x} = \phi + \psi \qquad \{3\}$$

SUMMARY OF SIEVERS' EQUATIONS

Hamilton's principle is used to derive the governing equations. They are,

$$-m\left(\ddot{y}+2v_O\dot{y}'+v_O{}^2y''\right)+\left(\frac{AG}{n}+T\right)y''-\frac{AG}{n}\phi'=0 \qquad \{4\}$$

$$J\left(\ddot{\phi}+2v_O\dot{\phi}'+v_O{}^2\phi''\right)+EI\phi''+\frac{AG}{n}(y'-\phi)=0 \qquad \{5\}$$

In this equation, y is the lateral deflection, m is the mass per unit of length, v_o is the transport velocity in the machine direction, v_y is the velocity of lateral deflection, J is the rotational inertia per unit of length, ϕ is the rotation of cross section, A is the cross sectional area, G is the shear modulus, n is the shear coefficient, E is Young's modulus and I is the area moment of inertia. The time derivatives in the equations have been transformed to a Eulerian frame of reference. The dot represents differentiation with respect t and a single quote mark indicates differentiation with respect to x.

THE STATIC EQUATION FOR WEB SHAPE

The leftmost acceleration terms in {4} and {5} can be eliminated because the imposed elastic deformations occur at frequencies that are too low for their inertial reactions to be significant.

$$\left(\frac{AG}{n}+T\right)y''-\frac{AG}{n}\phi' = 0 \qquad \{6\}$$

$$EI\phi'' + \frac{AG}{n}\left(y'-\phi\right) = 0 \qquad \{7\}$$

These equations can be reduced to a single expression involving only one dependent variable. Equation {6}is solved for ϕ' and then differentiated by x to produce values for ϕ'' and ϕ''' which are used to eliminate ϕ from {7}after it is differentiated once with respect to x. The result is the familiar equation,

$$\frac{d^4y}{dx^4}-K^2\frac{d^2y}{dx^2}=0 \qquad \{8\}$$

[1] Note: Sievers used the symbol θ to represent rotation of the cross section rotation (which she called face angle). In this paper ϕ is used, because θ is used in most of the literature as another way to represent $\partial y/\partial x$.

where,

$$K^2 = \frac{T}{EI\left(1 + \frac{nT}{AG}\right)} \qquad \{9\}$$

The solution to {8}will be familiar to anyone who has studied lateral dynamics.

$$y(x) = C_1 \sinh(Kx) + C_2 \cosh(Kx) + C_3 x + C_4 \qquad \{10\}$$

Static boundary conditions

The coefficients of {10}are determined from static boundary conditions. Four are needed. Lateral position at each end is an obvious choice for all three models and for a multi-span model it must obviously be continuous across rollers. That takes care of two conditions. For the Euler Bernoulli beam, slope at each end is the other logical choice because it will also be continuous across a roller. So, the Euler Bernoulli boundary conditions are,

$$\begin{array}{ll} y\big|_{x=0} = y_0 & y\big|_{x=L} = y_L \\ \left.\frac{dy}{dx}\right|_{x=0} = \theta_{w0} & \left.\frac{dy}{dx}\right|_{x=L} = \theta_{wL} \end{array} \qquad \{11\}$$

where L is the length of the span.

Timoshenko static boundary conditions

For the Timoshenko beam, the presence of shear makes it possible for slope to be discontinuous across a roller. So, it isn't preserved as the web passes from entry to exit. However, the cross section rotation is preserved, making it the logical choice for the other pair of boundary conditions.

To express the boundary conditions for the Timoshenko beam, expressions are needed for cross section rotation ϕ and shear angle ψ. First, differentiate {6} with respect to x and solve for ϕ''.

$$\frac{d^2\phi}{dx^2} = \frac{d^3y}{dx^3} a \qquad \{12\}$$

where

$$a = 1 + \frac{nT}{AG} \qquad \{13\}$$

Replacing $y' - \phi$ in {7}with ψ and using {12},

$$\psi = -EIa \frac{n}{AG} \frac{d^3y}{dx^3} \qquad \{14\}$$

Now, using {3},

$$\phi = \frac{dy}{dx} + EIa\,\frac{n}{AG}\,\frac{d^3y}{dx^3} \qquad \{15\}$$

The constant *EIan/AG* may be expressed as,

$$EIa\,\frac{n}{AG} = E\,\frac{W^2A}{12}\,\frac{n}{A\frac{E}{2(1+\mu)}}\,a = \frac{1}{6}W^2 n\left(1+\mu\right)a \qquad \{16\}$$

where W is the web width and μ is Poisson's ratio. So, ϕ becomes,

$$\phi = \frac{dy}{dx} + L^2 b\,\frac{d^3y}{dx^3} \qquad \{17\}$$

where,

$$b = \frac{1}{6}\,\frac{W^2}{L^2}\,n\left(1+\mu\right)a \qquad \{18\}$$

The constant in {16}has been divided by L^2 to nondimensionalize it.

Now, the Timoshenko beam boundary conditions can be defined. They are,

$$\begin{aligned} &y\big|_{x=0} = y_0 && y\big|_{x=L} = y_L \\ &\frac{dy}{dx}\bigg|_{x=0} + L^2 b\,\frac{d^3y}{dx^3}\bigg|_{x=0} = \phi_0 && \frac{dy}{dx}\bigg|_{x=L} + L^2 b\,\frac{d^3y}{dx^3}\bigg|_{x=L} = \phi_L \end{aligned} \qquad \{19\}$$

<u>Solving for the coefficients of {10}</u>

Differentiating {10}provides expressions for the derivatives in ϕ_0 and ϕ_L,

$$\frac{dy}{dx} = C_3 + C_1K\cosh(Kx) + C_2K\sinh(Kx) \qquad \{20\}$$

$$\frac{d^3y}{dx^3} = C_1K^3\cosh(Kx) + C_2K^3\sinh(Kx) \qquad \{21\}$$

Using {10} again for y_0 and y_L provides the four boundary conditions.

So, the boundary conditions of the Timoshenko beam model will be,

$$\begin{aligned} &\phi_0 = C_3 + C_1Ka \\ &\phi_L = C_3 + Ka\left(C_1\cosh(KL) + C_2\sinh(KL)\right) \\ &y_0 = C_2 + C_4 \\ &y_L = C_1\sinh(KL) + C_2\cosh(KL) + C_3L + C_4 \end{aligned} \qquad \{22\}$$

These are solved simultaneously for C_1, C_2, C_3 and C_4.

Inserting these values into {10}and collecting terms,

$$y(x) = y_0 + \left(y_0 - y_L\right)g_4(x,L) + \phi_L g_5(x,L) + \phi_0 g_6(x,L) \qquad \{23\}$$

where,

$$g_4(x) = [\cosh(Kx) + \cosh(KL) - \cosh(KL - Kx) - Kax \sinh(KL) - 1] / R$$

$$g_{5(x)} = [KLa(\cosh(Kx) - 1) - Kax(\cosh(KL) - 1) - \sinh(Kx)$$
$$- \sinh(KL - Kx) + \sinh(KL)] / KaR \quad \{24\}$$

$$g_6(x) = [\sinh(Kx) - \sinh(KL) + \sinh(KL - Kx) - KLa(\cosh(KL - Kx) - 1)$$
$$+ Ka(L - x)(\cosh(KL) - 1)] / KaR$$

and,

$$R = KLa \sinh(KL) - 2(\cosh(KL - 1) \quad \{25\}$$

Following the example of Young, Shelton and Kardimilas (YSK) [3], y_0 appears twice in expression {23}. This reduces the number of shape factors from four to three.

A FORK IN THE ROAD

It is at this point where the analysis diverges from the approach used by Sievers. In her method it is unnecessary to deal with the details of ϕ. However, to get a solution in terms of transfer functions, ϕ must be expressed in terms of y_L and its derivatives.

The differences are illustrated graphically in the block diagrams of Figure 4.

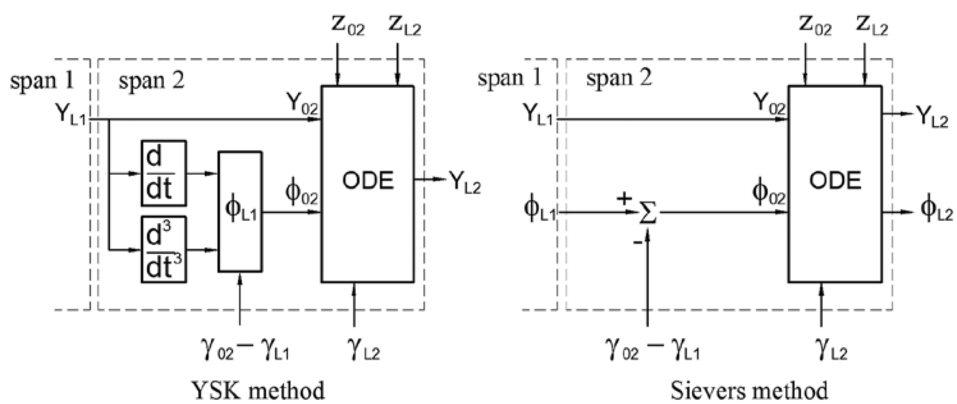

Figure 4
Graphic illustrations of the YSK and Sievers methods

In the YSK method, only the lateral displacement is passed to the next span. Since all of the information about its derivatives is implicit in the lateral displacement, it is possible to reconstruct the cross section rotation from the previous span by mathematical manipulation. This may seem redundant, but it has the advantage of making it possible to describe the system with transfer functions.

THE YSK METHOD

The goal is to create a time-based O. D. E. that can be solved for y_L. It seems reasonable to start with a spatial derivative of equation {23} and find a way to replace spatial derivatives of y_L with time derivatives. The first spatial derivative of {22}isn't suitable because two of the main terms are zero at $y = L$. The next best choice is the second derivative.

$$\frac{dy_L^2}{dx^2} = \left(y_0 - y_L\right)\frac{g_1}{L^2} + \phi_L\,\frac{g_2}{L} + \phi_0\,\frac{g_3}{L} \qquad \{26\}$$

Where,

$$\begin{aligned} g_1 &= L^2\left(g_4''(L)\right) = \frac{K^2L^2a\left(\cosh(KL)-1\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]} \\ g_2 &= L\left(g_5''(L)\right) = \frac{KL\left(KLa\cosh(KL)-\sinh(KL)\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]} \\ g_3 &= L\left(g_6''(L)\right) = \frac{KL\left(\sinh(KL)-KLa\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]} \end{aligned} \qquad \{27\}$$

Using {17},

$$\phi_L = \frac{dy_L}{dx} + L^2 b\,\frac{d^3y_L}{dx^3} \qquad \{28\}$$

and,

$$\phi_0 = \frac{dy_0}{dx} + L^2 b\,\frac{d^3y_0}{dx^3} \qquad \{29\}$$

Using {28} and {29} , equation {26} can be converted into a relationship involving only $y_{L,}$, y_0 and their spatial derivatives.

$$\frac{dy_L^2}{dx^2} = \left(y_0 - y_L\right)\frac{g_1}{L^2} + \left(\frac{dy_L}{dx} + L^2 b\frac{d^3y_L}{dx^3}\right)\frac{g_2}{L} + \left(\frac{dy_0}{dx} + L^2 b\frac{d^3y_0}{dx^3}\right)\frac{g_3}{L} \qquad \{30\}$$

<u>Replacement of spatial derivatives with time derivatives</u>

Relationships are needed between spatial and time derivatives. The normal entry rule provides one of them. It is,

$$\frac{\partial y_L}{\partial t} = v_o\left(\gamma_L - \left.\frac{\partial y_L}{\partial x}\right|_{x=L}\right) \qquad \{31\}$$

Adding the velocity of the roller

$$\frac{dy_L}{dt} = v_o \left(\gamma_L - \left.\frac{\partial y_L}{\partial x}\right|_{x=L} \right) + \frac{dz_L}{dt} \tag{32}$$

where v_o is the transport velocity of the web, γ_L is the angular misalignment of the roller, and z_L is the lateral velocity of the roller relative to ground (it may be a web guide roller). Solving {32}for $\partial y_L/\partial x$,

$$\left.\frac{\partial y_L}{\partial x}\right|_{x=L} = \frac{1}{v_o}\left(\frac{dz_L}{dt} - \frac{dy_L}{dt} \right) + \gamma_L \tag{33}$$

Using the chain rule, the second spatial derivative is,

$$\frac{d^2 y}{dt^2} = \frac{\partial^2 y(x,t)}{\partial t^2} + 2\frac{\partial y(x,t)}{\partial x \partial t} v_o + \frac{\partial^2 y(x,t)}{\partial x^2} v_o{}^2 = 0 \tag{34}$$

The cross derivative can be eliminated by taking the spatial derivative of{31},

$$\frac{\partial^2 y_L}{\partial x \partial t} = -v_o \left.\frac{\partial^2 y_L}{\partial x^2}\right|_{x=L} \tag{35}$$

Substituting {35}in{34},

$$\frac{d^2 y_L}{dt^2} = v_o{}^2 \left(\left.\frac{\partial y_L^2}{\partial x^2}\right|_{x=L} \right) \tag{36}$$

Note: *In the steady state, the time derivative on the left goes to zero and {36} becomes a statement of the steady state 4th boundary condition discovered by Shelton*.

A similar procedure for the third derivative yields,

$$\frac{d^3 y_L}{dt^3} = -v_o{}^3 \left(\left.\frac{\partial^3 y_L}{\partial x^3}\right|_{x=L} \right) \tag{37}$$

Note that the signs in{31}, {36}and{37} alternate.

Adding the acceleration and jerk for the rollers,

$$\begin{aligned} \frac{d^2 y_L}{dt^2} &= v_o{}^2 \left(\left.\frac{\partial y_L^2}{\partial x^2}\right|_{x=L} \right) + \frac{dz_L^2}{dt^2} \\ \frac{d^3 y_L}{dt^3} &= v_o{}^3 \left(\left.\frac{\partial y_L^3}{\partial x^3}\right|_{x=L} \right) + \frac{dz_L^3}{dt^3} \end{aligned} \tag{38}$$

Solving {38} for the spatial derivatives yields,

$$\frac{d^2y_L}{dx^2}=\frac{1}{v_o{}^2}\left(\frac{d^2y_L}{dt^2}-\frac{d^2z_L}{dt^2}\right)$$
$$\frac{d^3y_L}{dx^3}=\frac{1}{v_o{}^3}\left(\frac{d^3z_L}{dt^3}-\frac{d^3y_L}{dt^3}\right) \qquad \{39\}$$

Substituting{33} and {39}into{30},

$$\frac{dy_L^2}{dx^2} = \left(y_0-y_L\right)\frac{g_1}{L^2}+\left[\frac{1}{v_o}\left(\frac{dz_L}{dt}-\frac{dy_L}{dt}\right)+\gamma_L+L^2b\frac{1}{v_o{}^3}\left(\frac{dz_L^3}{dt^3}-\frac{d^3y_L}{dt^3}\right)\right]\frac{g_2}{L}$$
$$+\left(\frac{dy_0}{dx}+L^2b\frac{d^3y_0}{dx^3}\right)\frac{g_3}{L} \qquad \{40\}$$

A basic assumption in the Timoshenko multi-span model is that ϕ_0 at the exit of the upstream roller is equal to the value of the ϕ_L at the entry of that roller (with the exception of the effect of γ, the roller angle). Therefore, replacing the spatial derivatives with time derivatives in the same way as for y_L will recreate ϕ_L from the previous span and fulfill the multi-span requirement. Note: The fact that the factor L^2 in the ϕ_0 term applies to the current span rather than the previous span does not ruin this relationship because it is cancelled by a factor of L^2 in the denominator of the expression for b. The equation for the span, then becomes,

$$\frac{1}{v_o{}^2}\left(\frac{d^2y_L}{dt^2}-\frac{d^2z_L}{dt^2}\right) = \left(y_0-y_L\right)\frac{g_1}{L^2}+\left[\frac{1}{v_o}\left(\frac{dz_L}{dt}-\frac{dy_L}{dt}\right)+\gamma_L+L^2b\frac{1}{v_o{}^3}\left(\frac{d^3z_L}{dt^3}-\frac{d^3y_L}{dt^3}\right)\right]\frac{g_2}{L}$$
$$+\left[\frac{1}{v_o}\left(\frac{dz_0}{dt}-\frac{dy_0}{dt}\right)+\gamma_0+L^2b\frac{1}{v_o{}^3}\left(\frac{d^3z_0}{dt^3}-\frac{d^3y_0}{dt^3}\right)\right]\frac{g_3}{L}$$
{41}

There is one last issue to resolve in{41} . If the upstream roller is misaligned (either accidentally or because it is part of a guiding system), γ_0 will be nonzero and will not be the same as γ_L in the previous span.

Effect of roller axis angle on γ

The variable γ_L is the projection of roller alignment onto the plane of the web. Assuming that the angular motion of the roller is in a plane that is parallel to the cross machine direction,

$$\gamma_{L1} = \gamma\cos(\alpha) \quad \text{and} \quad \gamma_0 = -\gamma\cos(\beta) \qquad \{42\}$$

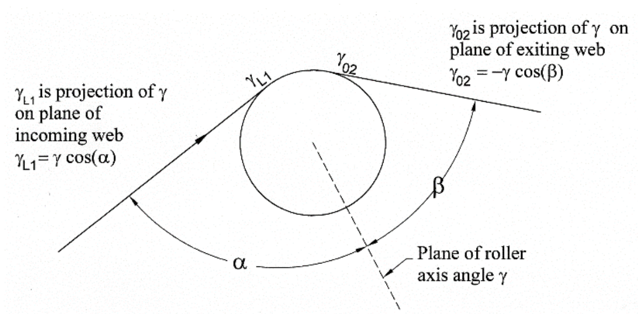

Figure 5

Relationship of plane of roller axis motion and web planes

The transfer function

The next step is to take the Laplace transform of equation {41}with all the initial conditions equal to zero. Equation {43}is the final result of collecting terms and substituting τ for L/V.

There are several ways to check this result.

1. If n is set to zero, it should become exactly equivalent to the Euler Bernoulli model in the YSK paper (also described nicely in a later paper by Pagilla and Seshradi [6]). Note: When $n = 0$, then $b = 0$ and $a = 1$.
2. When the only inputs are z_0 and z_L with $\gamma_L = \gamma_0 = z_l/L$, corresponding to a displacement guide, the transfer function should become unity because the web and rollers move as a rigid body.
3. When the only input is z_L with $\gamma_L = z_l/L1$, with the pivoting radius, $L1$, of the roller equal to $(Kc)L$, the static gain of the transfer function should be unity. This corresponds to a remote pivot steering guide with neutral steering. The curvature factor, Kc is defined in {1}and was derived by Shelton[2] for a static single-span Timoshenko model. Shelton's equation was confirmed for the modified Sievers model in a companion paper [4] presented at this conference.

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[2] The equation published by Shelton on page 64 of his dissertation (2.4.17) is incorrect. He informed me of the corrections in a private communication in the late '90s. The corrected equation for K_c is number {64} in Appendix A of this paper.

$$y_L = y_0 \frac{\left(-\tau b g_3 s^3 - \frac{g_3}{\tau}s + \frac{g_1}{\tau^2}\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)} + z_L \frac{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau}s\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)}$$

$$+ z_0 \frac{\left(\frac{g_3}{\tau}s + \tau b g_3 s^3\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)} + \gamma_L \frac{\left(\frac{v_o}{\tau} g_2\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)} \qquad \{43\}$$

$$+ \gamma_0 \frac{\left(g_3 \frac{v_o}{\tau}\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)}$$

<u>Test 1 (defaults to Euler Bernoulli when n = 0) - pass</u>

If $n = 0$ and $b = 0$, the Euler Bernoulli model is,

$$y_L = y_0 \frac{\left(-\frac{g_3}{\tau}s + \frac{g_1}{\tau^2}\right)}{\left(s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)} + z_L \frac{\left(s^2 + \frac{g_2}{\tau}s\right)}{\left(s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)} + z_0 \frac{\left(\frac{g_3}{\tau}s\right)}{\left(s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)}$$

$$+ \gamma_L \frac{\left(\frac{v_o}{\tau} g_2\right)}{\left(s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)} + \gamma_0 \frac{\left(g_3 \frac{v_o}{\tau}\right)}{\left(s^2 + \frac{g_2}{\tau}s + \frac{g_1}{\tau^2}\right)} \qquad \{44\}$$

This matches equation 11 for the Bernoulli beam model described in the Pagilla Seshradi paper [6] and the values of g_1, g_2, and g_3 also match the corresponding constants f_1, f_2 and f_3 when $a = 1$.

<u>Test 2 (displacement guide with z input has no dynamics) - pass</u>

When $\gamma_L = \gamma_0 = z_L/L$.

$$y_L = z_L \frac{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{1}{L}\frac{v_o}{\tau} g_2 + \frac{1}{L} g_3 \frac{v_o}{\tau}\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{g_1}{\tau^2}\right)} = z_L \frac{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{g_2}{\tau^2} + \frac{g_3}{\tau^2}\right)}{\left(\tau g_2 b s^3 + s^2 + \frac{g_2}{\tau} s + \frac{g_1}{\tau^2}\right)} \qquad \{45\}$$

It can be shown that $g_2 + g_3 = g_1$. So, $y_L = z_L$.

Test 3 (curvature factor = static gain) – fail

When γ_L in equation {43} is equal to $z_L/(K_c L)$, and all other inputs are zero, the static gain for the transfer function should be unity and K_c will be,

$$\tilde{K}_c = \frac{g_2}{g_1} \qquad \{46\}$$

Substituting values from {27},

$$\tilde{K}_c = \frac{\left(KLa\cosh(KL) - \sinh(KL)\right)}{KLa\left(\cosh(KL) - 1\right)} \qquad \{47\}$$

This disagrees with two other sources.

1. The static gain calculated for the modified Sievers Timoshenko beam [4], (equation 59). It is repeated below in {48}.
2. The value calculated for a static Timoshenko beam by Shelton {64}.

It is interesting to note that {47}reduces to the correct value for the Euler Bernoulli beam, when $a = 1$.

The difference appears slight. The constant, *a*, has only to be moved inside the parenthesis to get the correct value. This immediately raises the suspicion that it is an algebraic error. If that is the case, the author couldn't find it. In any event, the correct value for K_c is,

$$K_c = \frac{KLa\cosh(KL) - \sinh(KL)}{KL\left(a\cosh(KL) - 1\right)} \qquad \{48\}$$

For a beam with shear, if *KL* is less than 1, there is a big difference between $\tilde{K}_c$ and K_c . Curvature factors for the following parameters are $K_c = 0.89$ and $\tilde{K}_c = 2.8$.

L = 22 inches	T = 44.5 lbf
h = 0.0034 inches	E = 550,000 psi
W =44.5 inches	

It is significant that K_c, as derived from the modified Sievers Timoshenko model is correct, yet that model is based on the same governing equations as this one. The main difference is that the cross section rotations are not expressed in terms of the spatial derivatives of *y*.

Shelton's dynamic Timoshenko model

There is one last point on this topic that deserves mention. In his dissertation, Shelton develops a dynamic single span model that includes shear. As part of that model he develops a relationship for the curvature factor. It is based on the assumption that the acceleration equation (serving the same purpose as the acceleration equation in{36}) has an extra term due to shear. The equation is,

$$\frac{d^2 y_L}{dt^2} = v_o{}^2 \left(\left. \frac{\partial^2 y}{\partial x^2} \right|_{x=L} \right) + \frac{d^2 z}{dt^2} - v_o \frac{d\theta_{Ls}}{dt} \qquad \{49\}$$

Where θ_{Ls} is the angle of shear (a possible reason for including the shear term is discussed in another paper [1]). This leads to a value for the curvature factor that is worse than{47}. It is,

$$\hat{K}_c = \frac{\hat{f}_2}{\hat{f}_1} \qquad \{50\}$$

where,

$$\hat{f}_1 = \left(K_e L\right)^2 \frac{\sinh\left(K_e L\right)}{K_e L\left(\cosh\left(K_e L\right)+1+\frac{1}{a}\frac{nT}{AG}\left(\cosh\left(K_e L\right)+1\right)-2\sinh\left(K_e L\right)\right)} \qquad \{51\}$$

$$\hat{f}_2 = K_e L \frac{K_e L\cosh\left(K_e L\right)-\sinh\left(K_e L\right)+\left(nT/AG\right)\left(2K_e L-\sinh\left(K_e L\right)\right)}{K_e L\sinh\left(K_e L\right)-2\left(\cosh\left(K_e L\right)-1\right)+\left(nT/AG\right)\left(\cosh\left(K_e L\right)-1\right)} \qquad \{52\}$$

Using the same data that produced $\tilde{\hat{K}}_c = 2.8$, produces a value for $\hat{K}_c$ of *5.2.*

Shelton used this dynamic shear model to derive transfer functions for response at a fixed idler and for a steering guide, but he must have had concerns about their validity because he didn't produce plots of their frequency response as he did for everything else.

Until the curvature factor issue is resolved, I would not advise using the model described in this section.

HOW MUCH DIFFERENCE DOES SHEAR MAKE?

Which Timoshenko model ?

The only dynamic lateral model with shear that I trust is a modification of one based on Sievers' work [1]. It passed all three of the tests described above. Sievers' original model and the modified model are described in the companion to this paper [4]. The original model won't do because it can't handle misaligned rollers.

The O. D. E. for the modified Sievers model is,

$$\frac{d^2 y_L}{dt^2} = \left(y_0 - y_L\right)\frac{v_o{}^2}{L^2}\left(g_1 - \frac{g_2 h_1}{h_2}\right) + \frac{g_2}{h_2}\left[\frac{v_o}{L}\left(\frac{dz_L}{dt} - \frac{dy_L}{dt}\right) + \frac{v_o{}^2}{L}\gamma_L\right] + \frac{v_o{}^2 \phi_0}{L}\left(g_3 - \frac{g_2 h_3}{h_2}\right) + \frac{d^2 z_L}{dt^2} \qquad \{53\}$$

where,

$$g_1 = L^2\left(g_4''(L)\right) = \frac{K^2L^2a\left(\cosh(KL)-1\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$g_2 = L\left(g_5''(L)\right) = \frac{KL\left(KLa\cosh(KL)-\sinh(KL)\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]} \qquad \{54\}$$

$$g_3 = L\left(g_6''(L)\right) = \frac{KL\left(\sinh(KL)-KLa\right)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

and,

$$h_1 = L\left(g_4'(L)\right) = \frac{KLa\sinh(KL)(1-a)}{a\left[KLa\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

$$h_2 = g_5'(L) = \frac{(a+1)\left(1-\cosh(KL)\right)+KLa\sinh(KL)}{a\left[KLA\sinh(KL)-2\left(\cosh(KL)-1\right)\right]} \qquad \{55\}$$

$$h_3 = g_6'(L) = \frac{(a-1)\left(1-\cosh(KL)\right)}{a\left[KLA\sinh(KL)-2\left(\cosh(KL)-1\right)\right]}$$

Equation {53}can only be solved as part of a set of simultaneous equations that include the starting point where both y_L and ϕ_L are known.

Typical data

An example of the difference between the Timoshenko and Euler Bernoulli models is shown in Figure 6.

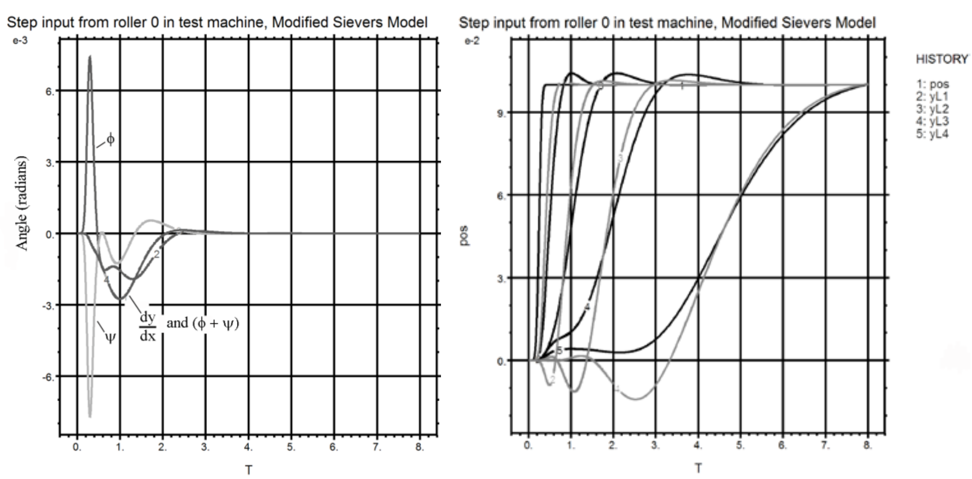

Figure 6

Comparison of Timoshenko and Euler Bernoulli models for step input

The model parameters are,

v_o = 200 fpm	T = 44.5 lbf
h = 0.0034 inches	E = 550,000 psi
W =44.5 inches	μ = 0.3

The four spans have lengths of 10, 22, 40 and 120 inches. The right hand graph in Figure 6 is a plot of the lateral displacements at the downstream ends of each one. The dark lines are for the Timoshenko model and the gray lines are for the Euler Bernoulli model. The leftmost curve is the input displacement at the upstream roller of the first span.

The left hand graph in Figure 6 shows ϕ, ψ and dy_L/dx for the downstream end of the second span. The sum of ϕ and ψ is also plotted, but isn't visible because it is identical to the curve for dy_L/dx (as it should be). It is clear from Figure 6 that for this span, ϕ, ψ and dy_L/dx are comparable to one another in magnitude.

It's interesting that the Euler Bernoulli model exhibits nonminumum phase behavior (initially going in the wrong direction), but the Timoshenko model doesn't. I have received several anecdotal accounts of incidental observations of nonminimum phase behavior during lab experiments.

TRACTION

All of the simulations and analysis described here assume that the web becomes locked to a roller surface at the line of entry and stays locked until it reaches the exit, but we all know that this isn't true – especially during transient conditions. It was for this reason that I was frankly surprised that Sievers model worked as well as it did. This is an area of web handling that needs more attention.

CONCLUSIONS

- A YSK-type Timoshenko model has been developed, and it looks quite plausible. However, it produces a value for the curvature factor that doesn't make sense. After exhaustive troubleshooting, I've concluded that the problem is most likely something of a conceptual nature.
- The modified Sievers' model was used to compare multi-span models with and without shear. The simulations indicate that there are nontrivial differences.
    - In short spans ϕ, ψ and dy_L/dx can be comparable to one another in magnitude.
    - For short spans there are significant qualitative differences in response to a step input.
    - The optimum time response for a remote pivot steering guide occurs when the radius of rotation of the guide mechanism equals K_cL. In short spans shear significantly affects K_c.

- Figure 6 is suggestive of a straightforward experiment which could shed considerable light on lateral dynamics models. It would be a simple matter to arrange a displacement guide to produce a step input into a series of spans instrumented to monitor their displacements.

APPENDIX A

The following equations are corrections to his dissertation [5] communicated to me over ten years ago by John Shelton.

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$$N_L + T\theta_L = -TK_e C_1 \qquad \{56\}$$

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$$C_3 = -C_1 K_e a \qquad \text{where} \quad a = 1 + \frac{nT}{AG} \qquad \{57\}$$

$$C_2 = \frac{\theta_L - C_1 K_e \left[\cosh(K_e L) - a\right]}{K_e \sinh(K_e L)} \qquad \{58\}$$

$$C_1 = -\frac{\theta_L}{K_e}\left[\frac{\cosh(K_e L)}{a\cosh(K_e L) - 1}\right] \qquad (2.4.12) \qquad \{59\}$$

$$C_2 = \frac{\theta\, L}{K_e}\left[\frac{\sinh(K_e L)}{a\cosh(K_e L) - 1}\right] \qquad (2.4.13) \qquad \{60\}$$

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$$C_3 = \theta_L \left[\frac{a\cosh(K_e L)}{a\cosh(K_e L) - 1} \right] \qquad (2.4.14) \qquad \{61\}$$

$$C_4 = \frac{\theta_L}{K_e} \left[\frac{-\sinh(K_e L)}{a\cosh(K_e L) - 1} \right] \qquad (2.4.15) \qquad \{62\}$$

$$\frac{y}{L} = \theta_L \left[\begin{array}{l} \frac{a\cosh(K_e L)}{a\cosh(K_e L)-1}\left(\frac{x}{K_e L}\right) - \left(\frac{1}{K_e L}\right)\frac{\cosh(K_e L)}{a\cosh(K_e L)-1}\sinh(K_e x) \\ +\left(\frac{1}{K_e L}\right)\frac{\sinh(K_e L)}{a\cosh(K_e L)-1}\left(\cosh(K_e x)-1\right) \end{array} \right] \qquad (2.4.16) \qquad \{63\}$$

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$$\frac{y_L}{L\theta_L} = K_c = \frac{1}{K_e L}\frac{\sinh(K_e L) - aK_e L\cosh(K_e L)}{1 - a\cosh(K_e L)} \qquad (2.4.17) \qquad \{64\}$$

$$M_o = -EI\theta_L K_e \left[\frac{\sinh(K_e L)}{a\cosh(K_e L) - 1} \right] \qquad \{65\}$$

$$\frac{M_o}{Ty_L} = -\frac{1}{K_e L K_c} \left[\frac{\sinh(K_e L)}{a^2\cosh(K_e L) - a} \right] \qquad (2.4.18) \qquad \{66\}$$

CORRECTIONS

1. Removed superfluous n/AG term in equations (19)
2. Replaced partial with total derivatives in equations (39) and (41).
3. Denominators in 39 ii and 40 changed from dt^2 to dt^3.
4. Corrected γ_{02} Figure 5

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[1] Sievers, L., "Modeling and Control of Lateral Web Dynamics", PhD Thesis, Rensselaer Polytechnic Institute, Troy, NY, 1987

[2] Sievers, L., Balas, M. K., Flowtow, A., "Modeling of Web Conveyance Systems for Mutivariable Control", IEEE Transactions of Automatic Control, Vol. 33, No. 6 June 1988

[3] Young, G. E., Shelton, J. J., and Kardamilas, C. E., "Modeling and Control of Multiple Web Spans Using State Estimation", ASME J. of Dynamic Systems, Measurement and Control, 111, pp 505-510

[4] Brown, J. L., "A Belated Appreciation of Lisa Sievers' Thesis", Proceedings of the Thirteenth International Web Handling Conference", June 2015

[5] Shelton, J. J., “Lateral Dynamics of a Moving Web”, PhD Thesis, Oklahoma State University, July 1968
[6] Seshradi, A., Pagilla, P. R., “Optimal Web Guiding”, Journal of Dynamic Systems, Measurement, and Control, January 2010, Vol. 132